Chapter 2

Schematic Models of Constitutive Behavior of Solid Materials

1 Introduction

Much can be learned about the deformation behavior of materials using the methods of continuum mechanics. While there is rich diversity of structure at various levels in any piece of engineering material, a fundamental assumption of continuum theory is to assume materials consist of aggregates of representative volume elements (RVE). The deformation response of the material is then the result of the collective response of the RVEs. This is what is known as the constitutive behavior of the material. Such volume elements are large enough that material inhomogeneities can be ignored and average material property values can be used, but also small enough that it is entirely legitimate to apply the methods of mathematical analysis. In the case of engineering metals and alloys, representative volume elements of a fraction of a millimeter in size are adequate while in coarser grained materials such as concrete appropriate sizes are of the order of a few centimeters. This chapter provides an overview of schematic models of the most common types of materials behavior.

2 Mechanical Testing of Materials

Laboratory tests are widely used to investigate the constitutive behavior of materials. Mechanical and thermal testing are routinely used in thermomechanics. Most testing procedures are fully standardized and require carefully prepared specimens. Robust testing machines equipped with accurate devices for measurement and control of forces, displacements, temperatures, crack lengths, etc. are readily available commercially. Commonly used mechanical tests include

- Tension Test,
- Compression Test,
- Creep Test (constant stress),

- Relaxation Test (constant strain),
- Multiple Hardening-Relaxation Test,
- Cyclic Tests (periodic load),
- Fracture and Fatigue Tests,
- Multi axial Loading Tests.

The results of mechanical testing are often combined with simplified models of material behavior to produce specific expressions to be used in analysis and design by using an appropriate least squares procedure. However, there is a large number of sources of variability in mechanical testing experiments leading to significant scattering in the results and this needs to be taken also into account.

3 Schematic Models of Constitutive Behavior

Analogical models have long been used to produce schematic representations of the mechanical behavior of various materials. These models consists of simple mechanical elements displaying the characteristic property of interest. Specifically, a spring is used to represent linear elasticity, a piston immersed in a viscous fluid (damper) represents viscosity and a skidding block may represent plasticity. Complex mechanical behavior can also be represented using the above elements by combining them into series or parallel networks.

The limiting cases of material behavior are the rigid solid and the ideal fluid. A rigid solid moves as a whole and without changing shape under load while an ideal fluid deforms no matter how small the load applied to it. Most real materials of interest in thermomechanics exhibit behavior between these two extremes. Following are some examples.

3.1 The Linear Elastic Solid

A linear elastic solid can be represented using a spring element. The resulting constitutive equation relating stress to strain is of the form

$$\sigma = E\epsilon$$

The linear elastic solid exhibits instantaneous reversibility of the deformation. Alternatively, in a viscoelastic solid, the recovery of the deformation is delayed. An appropriate model is obtained by connecting a spring and a damper in parallel (Kevin-Voigt model). The strain-time relationship corresponding to a creep test under stress σ_0 at $t = 0$ is

$$\epsilon = \frac{\sigma_0}{E}[1 - \exp(-\frac{E}{\eta}t)]$$

where E the elastic modulus and η the viscosity.

3.2 The Viscoelastic Solid

A viscoelastic solid may be represented by a model consisting of a spring and a damper connected in series (Maxwell model). Under stress σ, the constitutive response of the Maxwell model is

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

The stress-time relationship corresponding to a relaxation test under strain ϵ_0 at $t = 0$ is

$$\sigma = E\epsilon_0 \exp(-\frac{E}{\eta}t)$$

3.3 The Plastic Solid

A plastic solid exhibits instantaneous permanent deformation that remains even upon removal of the load. Various possible models exist. A rigid, perfectly plastic solid does not deform until a specific stress threshold σ_s is reached. Subsequently, the strain becomes arbitrary. Alternatively, a linear elastic-perfectly plastic solid deforms elastically for stresses below the threshold and the strain becomes arbitrary once σ_s is reached. This behavior can be represented by a model consisting of a spring and a skidding block connected in series (Saint-Venant model).

In an elastoplastic hardening solid the total strain is due to an elastic contribution and a plastic contribution. The latter is zero when $\sigma < \sigma_s$. I.e.

$$\begin{aligned} \epsilon = \epsilon_e &= \frac{\sigma}{E} \quad \text{if } |\sigma| < \sigma_s \\ \epsilon = \epsilon_e + \epsilon_p &= \frac{\sigma}{E} + g(\sigma) \quad \text{if } |\sigma| \geq \sigma_s \end{aligned}$$

where $g(\sigma)$ is the function describing the hardening effect. A modified Saint-Venant model consisting of multiple series spring-skidding block couples arranged in parallel, with the various couples exhibiting monotonically increasing threshold stresses.

3.4 The Creeping, Viscoplastic Solid

Viscoplastic solids deform permanently under load, just like plastic solids do but the strain continues as a function of time (creep) even at constant load. In a perfect viscoplastic solid, the constitutive equation is stated as a relationship between stress and strain rate; an example is the power law

$$\sigma = \lambda \dot{\epsilon}^{1/N}$$

Alternatively, the elastic-perfectly viscoplastic solid exhibits linear elastic behavior at low stresses, i.e.

$$\epsilon = \epsilon_e = \frac{\sigma}{E} \quad \text{if } |\sigma| < \sigma_s$$

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_p = \frac{\dot{\sigma}}{E} + f(\sigma) \quad \text{if } |\sigma| \geq \sigma_s$$

The above behavior may be represented schematically by a model consisting of a non-linear damper and a skidding block in parallel, connected to a spring in series (Bingham-Norton model). Finally, the elastoviscoplastic hardening solid deforms elastically at low loads but the stress subsequently becomes a function of the strain and the strain rate.

In cases involving hardening, the increased resistance to deformation with increased strain must be accounted for. The simplest approach is to assume that the elastic domain in stress space for the deforming material expands uniformly in all directions (isotropic hardening). Alternatively, the elastic domain in stress space may remain constant in size but it translates through it (kinematic hardening). This provides a representation of the Bauschinger effect.

Various possibilities occur under cyclic loading experiments (controlled strain or controlled stress). Cyclic hardening or cyclic softening of the material may be observed depending on the nature of the material, the temperature and the initial state. Such behavior may become stabilized after a large enough number of cycles. Shakedown, ratchetting and relaxation can also be observed.

3.5 Damage, Fracture, Contact and other Effects

The determination of fracture and fatigue behavior of materials requires specialized observation methods (fractography). Material volume elements are examined for damage (microductility, microdecohesion, microcracking). The fracture behavior may be preceeded by negligible plastic strain (brittle fracture) or follow after significant permanent deformation (ductile fracture). Fracture studies often also involve determination of the growth rate of cracks in the material. The peculiarities of crack growth phenomena are typically very dependent on the material nature, temperature and the type of loading.

In situations involving deforming materials in contact with other solids, friction phenomena are involved. According to the Coulomb model, a threshold level of stress is required to induce relative motion of two contacting solids.

In all of the above cases the deformation and fracture behavior may be further complicated by transient microstructural processes taking place inside the material.