Chapter 4

Linear Elasticity

1 Introduction

The simplest mechanical test consists of placing a standardized specimen with its ends in the grips of a tensile testing machine and then applying load under controlled conditions. Uniaxial loading conditions are thus approximately obtained. A force balance on a small element of the specimen yields the longitudinal (true) stress as

$$\sigma = \frac{F}{A}$$

where F is the applied force and A is the (instantaneous) cross sectional area of the specimen. Alternatively, if the initial cross sectional area A_0 is used, one obtains the engineering stress

$$\sigma_e = \frac{F}{A_0}$$

For loading in the elastic regime, for most engineering materials $\sigma_e \approx \sigma$.

Likewise, the true strain is defined as

$$\epsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln(\frac{l}{l_0})$$

while the engineering strain is given by

$$\epsilon_e = \int_{l_0}^{l} \frac{dl}{l_0} = \frac{l - l_0}{l_0}$$

Again, for loading in the elastic regime, for most engineering materials $\epsilon_e \approx \epsilon$.

Linear elastic behavior in the tension test is well described by Hooke's law, namely

$$\sigma = E\epsilon$$

where E is the modulus of elasticity or Young's modulus. For most materials, this is a large number of the order of 10^{11} Pa.

Values of E can be readily determined by measuring the speed of propagation of longitudinal elastic waves in the material. Ultrasonic waves are induced by a piezoelectric device on the surface on the specimen and their rate of propagation accurately measured. The velocity of the longitudinal wave is given by

$$v_L = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}$$

Transverse wave propagation rates are also easily measured by ultrasonic techniques and the corresponding relationship is

$$v_T = \sqrt{\frac{G}{\rho}}$$

where G is the modulus of elasticity in shear or shear modulus.

The shear modulus is involved in the description of linear elastic behavior under shear loading, as encountered, for instance during torsion testing of thin walled pipes. In this case, if the shear stress is τ and the shear strain γ, Hooke's law is

$$\tau = G\gamma$$

2 Generalized Hooke's Law

The statement that the component of stress at a given point inside a linear elastic medium are linear homogeneous functions of the strain components at the point is known as the generalized Hooke's law. Mathematically, this implies that

$$\sigma^{ij} = D^{ijkl}\epsilon_{kl}$$

where σ^{ij} and ϵ_{ij} are, respectively the stress and strain tensor components. The quantity D^{ijkl} is the tensor of elastic constants and it characterizes the elastic properties of the medium. Since the stress tensor is symmetric, the elastic constants tensor consists of 36 components.

The elastic strain energy W is defined as the symmetric quadratic form

$$W = \frac{1}{2}\sigma^{ij}\epsilon_{kl} = \frac{1}{2}D^{ijkl}\epsilon_{ij}\epsilon_{kl}$$

and has the property that $\sigma^{ij} = \partial W/\partial\epsilon_{ij}$. Because of the symmetry of W, the actual number of elastic constants in the most general case is 21. This number is further reduced in special cases that are of much interest in applications. For instance, for isotropic materials (elastic properties the same in all directions) the number of elastic constants is 2. For orthotropic materials (characterized by three mutually perpendicular planes of symmetry) the number of constants is 9. If the material exhibits symmetry with respect to only one plane, the number of constants is 13.

3 Stress-Strain Relations for Isotropic Elastic Solids

The generalized Hooke's law for isotropic solids is

$$\sigma_{\alpha\alpha} = 3K\epsilon_{\alpha\alpha}$$
$$\sigma'_{ij} = 2G\epsilon'_{ij}$$

where K and G are the elastic constants bulk modulus and shear modulus, respectively and the primes denote the stress and strain deviators.

Combination of the above with the definition of stress and strain deviation tensors yields the following commonly used forms of Hooke's law; for stress, in terms of strain

$$\sigma_{ij} = \lambda\epsilon_{\alpha\alpha}\delta_{ij} + 2G\epsilon_{ij}$$

and for strain, in terms of stress

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{\alpha\alpha}\delta_{ij}$$

The constants λ and G are called Lame's constants, while E is Young's modulus and ν is Poisson's ratio. Any of the above elastic constants can be expressed in terms of the others and only two are independent. Values of the above elastic constants for a wide variety of engineering materials are readily available in handbooks.

For an isotropic elastic solid in a rectangular Cartesian system of coordinates, the constitutive equations of behavior then become

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]$$
$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})]$$
$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]$$

$$\epsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$
$$\epsilon_{yz} = \frac{1}{2G}\sigma_{yz}$$
$$\epsilon_{zx} = \frac{1}{2G}\sigma_{zx}$$

4 Stress-Strain Relations for Anisotropic Elastic Solids

It is conventional in studying elastic deformation of anisotropic materials to relabel the six stress and strain components as follows:

$$\sigma_{11} = \sigma_1$$

$$\sigma_{22} = \sigma_2$$
$$\sigma_{33} = \sigma_3$$
$$\sigma_{23} = \sigma_4$$
$$\sigma_{13} = \sigma_5$$
$$\sigma_{12} = \sigma_6$$

$$\epsilon_{11} = \epsilon_1$$
$$\epsilon_{22} = \epsilon_2$$
$$\epsilon_{33} = \epsilon_3$$
$$\epsilon_{23} = \frac{1}{2}\epsilon_4$$
$$\epsilon_{13} = \frac{1}{2}\epsilon_5$$
$$\epsilon_{12} = \frac{1}{2}\epsilon_6$$

With the new notation and using the summation convention, Hooke's law becomes

$$\sigma_i = C_{ij}\epsilon_j$$

or equivalently

$$\epsilon_i = S_{ij}\sigma_j$$

where C_{ij} and S_{ij} are, respectively the elastic stiffness and compliance matrices. Depending on the symmetries existing in the material, only a few components of the above matrices are nonzero. For instance, for single crystals with cubic structure only C_{11}, C_{12} and C_{44} are nonzero. Values of the components of the above matrices for a variety of anisotropic materials are readily available in handbooks.

5 Formulation of Linear Elastic Problems

For steady state conditions, the governing equations of the isotropic linear elastic solid are, the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = \sigma_{ij,j} + X_i = 0$$

the stress-strain relations

$$\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij}$$

and the small displacement, strain-displacement relations

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

Admissible strain fields are those that satisfy the equations of compatibility. The three equilibrium equations together with the six stress-strain relations constitute a set of nine equations for the nine unknowns u_i, σ_{ij}. One can show this system is complete, yields an unique solution under suitable boundary conditions and the resulting strain satisfies the compatibility relations.

The above system of equations must be solved in each particular case subject to appropriate boundary conditions. There are two fundamental boundary value problems of elasticity:

- Problem 1: Determine the stress and strain fields inside the elastic body subject to specified values of the displacement u_i at its boundary.

- Problem 2: Determine the stress and strain fields inside the elastic body subject to a specified values of the surface tractions T_i at its boundary.

Because of the nature of the boundary conditions associated with the two fundamental problems above, it is convenient to produce formulations of the elasticity problem involving only displacements or stresses.

Specifically., combining the strain-displacement relationships with Hooke's law and subsequent introduction of the result into the equilibrium equations yields the Navier equations

$$G u_{i,jj} + (\lambda + G) u_{j,ji} + X_i = 0$$

The above is then a set of three equations for the three unknowns u_i.

Alternatively, combination of the compatibility equations, Hooke's law and the equilibrium equations yields the Beltrami-Michell equations

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu} X_{k,k}\delta_{ij} - (X_{i,j} + X_{j,i})$$

An interesting special case of the equilibrium equations is obtained if the body force vanishes. In this case, one can readily show that the two invariants, the dilatation $e = \epsilon_{\alpha\alpha}$ satisfies

$$\nabla^2 e = 0$$

and the mean stress $\sigma = \frac{1}{3}\sigma_{\alpha\alpha}$ satisfies

$$\nabla^2 \sigma = 0$$

Furthermore

$$\nabla^4 u_i = 0$$

$$\nabla^4 \epsilon_{ij} = 0$$

$$\nabla^4 \sigma_{ij} = 0$$

i.e. the dilatation and the mean stress are harmonic functions while the displacement vector components, the strain tensor components and the stress tensor components are all biharmonic functions.

6 Strain Energy Function

An important quantity in continuum mechanics is the energy associated with deformation. The specific strain energy of the linear elastic material W was defined above as

$$W = \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl}$$

It has the properties

$$\frac{\partial W}{\partial \epsilon_{ij}} = \sigma_{ij}$$

and

$$\frac{\partial W}{\partial \sigma_{ij}} = \epsilon_{ij}$$

It can be shown that, as long as the strain energy function exists and is positive definite, the two fundamental boundary value problems of elasticity have unique solutions. Non-unique solutions may occur if W is not positive definite (e.g. buckling, inelastic deformation) or when the linear equations break down (e.g. finite deformation, forces with memory).

7 Potential Methods

Helmholtz' theorem states that any reasonable vector field $\mathbf{u}$ can be expressed in general in terms of potentials, i.e.

$$\mathbf{u} = \nabla \phi + \nabla \times \psi$$

where, here ϕ is the scalar potential and ψ is the vector potential. The scalar potential is directly related to the dilation e by

$$e = \phi, ii$$

while the vector potential relates to the rotation vector ω_i by

$$-2\omega_i = \psi_i, jj$$

A simple example of the use of potentials in elasticity is Lame's strain potential function ϕ defined by

$$2Gu_i = \frac{\partial \phi}{\partial x_i}$$

It can be shown that in the absence of body forces the potential is an harmonic function, i.e.

$$\nabla^2 \phi = 0$$

and the stress tensor components are given by

$$\sigma_{ij} = \phi_{,ij}$$

Since harmonic functions have long been studied and are well known, they can be readily used to obtain solutions to many practical problems.

8 Two-Dimensional Problems in Rectangular Cartesian Coordinates

In Cartesian coordinates (x, y), the Airy stress function $\Phi(x, y)$ is implicitly defined by the equations

$$\frac{\partial^2 \Phi}{\partial y^2} = \sigma_{xx} - V$$
$$\frac{\partial^2 \Phi}{\partial x \partial y} = -\sigma_{xy}$$
$$\frac{\partial^2 \Phi}{\partial x^2} = \sigma_{yy} - V$$

where the potential V is implicitly given by

$$\nabla V = -X$$

where X is the body force vector.

If $X = 0$ then

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^4} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

Hence, the stress function is a biharmonic function.

9 Two-Dimensional Problems in Polar Coordinates

The equations of equilibrium in two-dimensional polar coordinates (r, θ), where the body force acts only along the $r-$direction are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = 0$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} = 0$$

One can show that the appropriate form of the stress function equation required to satisfy compatibility is

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})(\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}) = 0$$

where

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r}$$

and

$$\sigma_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2}$$

An important special case is obtained when the stress distribution is symmetrical about an axis. The stress function equation in this case becomes

$$\frac{d^4 \Phi}{dr^4} + \frac{2}{r}\frac{d^3 \Phi}{dr^3} - \frac{1}{r^2}\frac{d^2 \Phi}{dr^2} + \frac{1}{r^3}\frac{d\Phi}{dr} = 0$$

One can show that a fairly general solution of the above is of the form

$$\Phi = A \log r + B r^2 \log r + C r^2 + D$$

10 Thermodynamics of Elastic Bodies

Thermodynamics accounts for the effects of thermal phenomena on the response of materials to mechanical loads. The first law of thermodynamics states that energy is conserved. The second law states that entropy is always created. The combined statement of the first and second laws of thermodynamics as applied to solid bodies is

$$dU = TdS + \frac{1}{\rho}\sigma_{ij} d\epsilon_{ij}$$

where U is the specific internal energy or energy per unit mass and S is the specific entropy or entropy per unit mass.

The Helmholtz and Gibbs free energy functions are often useful in solving practical problems. They are defined respectively as

$$F = U - TS$$

and

$$G = U - TS - \frac{1}{\rho}\sigma_{ij}\epsilon_{ij}$$

The equations of thermodynamics can be used to obtain expressions of the condition required for thermodynamic equilibrium. Two equivalent statements of the condition are

- A system is in thermodynamic equilibrium if for all possible variations of the state taking place at constant energy, the entropy change is negative, i.e. $(\Delta S)|_U < 0$, the entropy is maximal.
- A system is in thermodynamic equilibrium if for all possible variations of the state taking place at constant entropy, the energy change is non-negative, i.e. $(\Delta U)|_S \geq 0$, the energy is minimal (or positive definite).

The laws of thermodynamics impose definite restrictions on the mathematical form of constitutive equations. In the case of linear elastic bodies, combination of Hooke's law with the definition of strain energy yields

$$dW = \sigma_{ij}d\epsilon_{ij} = \lambda e de + 2G\epsilon_{ij}d\epsilon_{ij}$$

For a point in a body going from zero stress to a particular stressed state, the total strain energy becomes

$$W = \int dW = \frac{\lambda}{2}e^2 + G\epsilon_{ij}\epsilon_{ij}$$

or, in terms of the second invariant of the strain deviation tensor, $J_2 = \frac{1}{2}\epsilon'_{ij}\epsilon'_{ij}$,

$$W = \frac{1}{2}Ke^2 + 2GJ_2$$

Since, thermodynamics requires the energy to be positive definite, necessarily

$$E > 0$$

and

$$-1 < \nu < \frac{1}{2}$$

The strain energy function must be modified when the body is subjected to thermal loads in addition to mechanical ones since there is thermal expansion. The appropriate form of the strain energy in this case is

$$W = -\beta_{ij}(T - T_0)\epsilon_{ij} + \frac{1}{2}C'_{ijkl}\epsilon_{ij}\epsilon_{kl}$$

where β_{ij} is a tensor of thermal expansion coefficients and T_0 is a reference temperature. With this, the stress becomes

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}(T - T_0)$$

As in the isothermal case, the number of elastic constants is significantly reduced for an isotropic body and the expressions for stress, strain and energy become

$$\sigma_{ij} = \lambda e\delta_{ij} + 2G\epsilon_{ij} - \frac{E\alpha}{1-2\nu}(T - T_0)\delta_{ij}$$

$$\epsilon_{ij} = -\frac{\nu}{E}s\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij} + \alpha(T - T_0)\delta_{ij}$$

and

$$W = G[\epsilon_{ij}\epsilon_{ij} + \frac{\nu}{1-2\nu}e^2 - \frac{2(1+\nu)}{1-2\nu}\alpha(T - T_0)e]$$

where α is the coefficient of thermal expansion, $e = \epsilon_{ii}$ and $s = \sigma_{ii}$.

With the above, expressions for the various thermodynamic functions applicable to isotropic linear elastic solid can be derived. For instance, the internal energy is given by

$$U = \frac{1}{\rho}[\frac{1}{2}Ke^2 + 2GJ_2 + \frac{E\alpha}{1-2\nu}T_0 e + \rho(C_v)|_{e_{ij}=0}(T - T_0)]$$

and for the Gibbs free energy

$$G = \frac{1}{\rho}[\frac{\nu}{2E}\sigma^2 - \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \alpha(T - T_0)\sigma + \rho\int_{T_0}^{T} dT\int_{T_0}^{T}(C_p)|_{\sigma_{ij}=0}\frac{dT'}{T'}]$$

Moreover, various equations connecting the thermal and mechanical properties of the elastic solid are also readily derived. One example is the formula

$$C_p - C_v = \frac{T}{\rho}\frac{\partial\epsilon_{ij}}{\partial T}\frac{\partial\sigma_{ij}}{\partial T}$$

11 Thermoelasticity

When thermal energy is added to an elastic material it expands. For the simple unidimensional case of a bar of length L, initially at uniform temperature T_0 which is then heated to a nonuniform temperature T and thus grows in length by an amount ΔL, the relative uni-axial stretching due to thermal expansion is

$$\frac{\Delta L}{L} = \epsilon = \alpha(T - T_0)$$

where ϵ is the strain and α is the thermal expansion coefficient. For an isotropic cube of side L the (normal) thermoelastic strains are

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha(T - T_0)$$

It is conventional but not necessary to take $T_0 = 0$.

Since the heated region is joined to, and constrained by rigid surroundings, it can not expand freely but becomes subjected to compressive stresses. At the same time the colder portion is subjected to the pull exerted by of the adjacent hot portion and it is thus under tension. Although Hooke's law is still applicable, due account must be taken of the additional stresses created by thermal expansion.

The governing equations for the isotropic thermoelastic solid include the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = \sigma_{ij,j} + X_i = 0$$

where $i, j = 1, 2, 3$, the generalized thermoelastic stress-strain relations

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - \beta_{ij}(T - T_0) = \lambda\epsilon_{kk}\delta_{ij} + 2G\epsilon_{ij} - \beta\delta_{ij}\theta$$

where $\theta = T - T_0$ is the excess temperature distribution, and $\beta = \alpha E/(1 - 2\nu)$ where α is the thermal expansion coefficient.

Expressed as strain-stress relationships the above are

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{\mu\mu}\delta_{ij} + \alpha\theta\delta_{ij}$$

The small displacement strain-displacement relations are, as before

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Finally, the compatibility equations must also be satisfied.

The temperature distribution θ must be determined by solving the energy conservation equation

$$\frac{dU}{dt} = T\frac{\partial S}{\partial t} + \frac{1}{\rho}\sigma_{ij}V_{ij}$$

where U is the internal energy, S the entropy and

$$V_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$$

is the rate of deformation tensor where v_i is the velocity.

One can show that the following from the differential thermal energy balance equation can be derived from the above

$$\frac{\partial H}{\partial t} + \theta\beta_{ij}\frac{\partial \epsilon_{ij}}{\partial t} = \nabla \cdot (k\nabla\theta) + r$$

where H is the enthalpy, β_{ij} are experimentally determined numerical coefficients and r is the rate of internal energy generation.

The energy equation above must be solved subject to suitable boundary and initial conditions in order to determine the temperature field θ.

For steady state conditions in a medium of constant conductivity and without internal heat generation

$$\nabla^2\theta = 0$$

i.e. solutions to steady state heat conduction problems are harmonic functions.

In uncoupled, quasi-static thermoelastic theory, the mechanical coupling terms in the energy and the heat conduction equations are neglected. Therefore, the heat conduction problem and the thermoelastic deformation problem are handled separately.

By substituting the generalized thermoelastic stress-strain relations and the small displacement strain-displacement relations into the equilibrium equation one obtains the generalized Navier's equation

$$Gu_{i,\mu\mu} + (\lambda + G)u_{\mu,\mu i} + X_i - \beta\theta_{,i} = 0$$

The three thermomechanical equilibrium equations together with the energy equations and the six stress-strain relations constitute a set of ten equations for the ten unknowns u_i, τ_{ij} and θ. One can show this system is complete, yields an unique solution under suitable boundary conditions and the resulting strain satisfies the compatibility relations.

For the solution of thermoelastic problems Goodier introduced the displacement potential function ϕ as

$$\mathbf{u} = \nabla\phi = u_i = \frac{\partial\phi}{\partial x_i}$$

The above, when substituted into the generalized Navier equation and integrated yields

$$\phi_{,\mu\mu} = \frac{1}{\lambda + 2G}(P + \beta\theta)$$

where P is the potential for the assumed conservative body forces (i.e. $\mathbf{X} = -\nabla P$).

The solution of the above is the sum of a particular solution and the complementary solution of Laplace's equation ($\nabla^2\phi = 0$).

For plane strain conditions, on a $x - y$ plane in rectangular Cartesian coordinates, combination of the equilibrium equations and the compatibility condition yields

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \frac{\beta}{1-\nu}\nabla^2\theta$$

Introducing the stress function Φ defined by

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} + \beta\theta$$

$$\sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} + \beta\theta$$

$$\sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}$$

yields

$$\nabla^4\Phi = -\frac{\alpha E}{1-\nu}\nabla^2\theta$$

12 Finite Elements in Elasticity

Consider an elastic body being deformed by applied loads while constrained from rigid body motion. The stress principle of Euler and Cauchy states that when the body is in mechanical equilibrium the volume forces and surfaces at any small subdomain inside the body balance each other, i.e.

$$\int_\Omega f dv + \int_\Gamma t ds = 0$$

where f is the volume force vector field, t is the stress vector field and Ω and Γ are the volume and surface area of the subdomain.

Furthermore, Cauchy's theorem states that related to the stress vector t there is a stress tensor σ called the Cauchy stress tensor.

The displacement vector field $\mathbf{u}$ described the deformation everywhere inside the loaded body. However, it is convenient to introduce the strain tensor ϵ associated to the displacement in the small deformation regime characteristic of elastic phenomena

$$\epsilon_{ij} = \frac{1}{2}(\frac{\partial u}{\partial x_j} + \frac{\partial u}{\partial x_i}) = \nabla^s u$$

where ϵ_{ij} are the components of the strain tensor and ∇^s is called the symmetric gradient operator.

It is convenient in solid mechanics to state the condition of mechanical equilibrium in variational form. In this representation, the deformation state corresponding to mechanical equilibrium of the loaded body is the one that yields the minimum value of the functional

$$I = \int_V [\frac{1}{2}\epsilon : \sigma - f.u] + \int_{\Gamma_1} g.uds$$

where $\epsilon : \sigma = \sum_{ik} \epsilon_{ik}\sigma_{ik}$ This is just the expression of the principle of minimum energy. A closely related variational statement is the principle of virtual work.

All materials satisfy the above equation as a condition for mechanical equilibrium. However, depending on the material constitution, the response to a given load varies. This variation is described by means of constitutive equations of mechanical behavior. The constitutive equation for elastic behavior is Hooke's law

$$\epsilon = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}\sigma_{ii}I$$

where E is the elastic modulus, ν is Poisson's ratio. Sometimes, the shorthand notation $\sigma = C\epsilon$ is used instead.

Te commonly used displacement formulation of the finite element method as it is applied in solid mechanics is readily obtained by expressing the variational principle in terms of the displacement, i.e.

$$I = \int_V [\frac{1}{2}\nabla^s u : C\nabla^s u - f.u]dv + \int_{\Gamma_1} g.uds$$

In the end, the variational formulation leads to the fundamental equation of the finite element method, representing the condition of mechanical equilibrium between internal and external forces

$$\mathbf{Ku} = \mathbf{F}$$

where $\mathbf{K}$ is the stiffness matrix

$$\mathbf{K} = \sum_e \mathbf{A}^{eT}\mathbf{K}^e\mathbf{A}^e$$

with $\mathbf{A}^e$ is the Boolean matrix such that $A^e_{mn} = 1$ if the m-th elemental degree of freedom corresponds to the n-th global degree of freedom and zero otherwise. Here, $\mathbf{K}^e$ is the element stiffness matrix defined by

$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^{eT}\mathbf{C}\mathbf{B}^e dV$$

where $\mathbf{B}^e$ is the matrix containing the derivatives of the shape functions and C is the elasticity matrix containing the appropriate material properties. Moreover, $\mathbf{u}$ is the vector of nodal displacements and $\mathbf{F}$ is the vector of externally applied loads. The above system of linear algebraic equations can then solved using standard methods of numerical linear algebra once the stiffness matrix components are calculated by evaluating the stated integrals.

13 Examples

13.1 Torsion of Elastic Bars

Saint-Venant produced solutions for the torsion problem of a long bar (aligned with the $z-$direction), under small twist conditions by assuming the vector components of displacement to be given by

$$\begin{aligned} u &= -\alpha z y \\ v &= \alpha z x \\ w &= \alpha \phi(x, y) \end{aligned}$$

where α is the angle of twist per unit length of the bar and $\phi(x, y)$ is the warping function.

It can be shown that the warping function satisfies Laplace's equation

$$\nabla^2 \phi = 0$$

subject to the condition

$$\frac{\partial \phi}{\partial n} = y \cos(x, n) - x \cos(y, n)$$

at the lateral boundary of the bar and is therefore an harmonic function.

Alternatively, Prandtl proposed, in analogy with the stream function of hydrodynamics, the use of the stress function $\psi(x, y)$ defined by

$$\begin{aligned} \sigma_{xz} &= \frac{\partial \psi}{\partial y} \\ \sigma_{yz} &= -\frac{\partial \psi}{\partial x} \end{aligned}$$

It can be shown that the equations of elasticity for the torsion problem in terms of the stream function are equivalent to the problem of solving

$$\nabla^2 \psi = -2G\alpha$$

subject to

$$\psi = 0$$

on the boundary of the bar.

For instance for the bar of elliptical cross section (axes a and b), a suitable expression for the stress function is

$$\psi = k[(\frac{x}{a})^2 + (\frac{y}{b})^2) - 1]$$

With the above one can readily obtain closed form expressions for the shearing stresses τ_{zx}, τ_{zy} and $\tau_{z\alpha} = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$, the torque $M_t = 2\int\int \psi dxdy$, angle of twist α and the warpage w.

13.2 Bending of Beams

Consider a cantilever beam (length L, moment of inertia of the cross section I), loaded at one end with transverse force P. Select a rectangular Cartesian system of coordinates with the z-axis aligned with the axis of the beam and the $x-y$ plane coincident with its cross-section. The equilibrium equations are

$$\frac{\partial \sigma_{zx}}{\partial z} = 0$$
$$\frac{\partial \sigma_{yz}}{\partial z} = 0$$

and

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{Px}{I} = 0$$

It can be shown that in order to satisfy the compatibility requirement (expressed in terms of the stress tensor - Beltrami-Michell equations), the above problem can be represented by the following equivalent one in terms of the stress function ψ, namely, determine the function $\psi(x, y)$ satisfying

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = \frac{\nu}{1+\nu}\frac{Py}{I} - \frac{df}{dy}$$

subject to the condition $\psi = 0$ on the boundary of the cross section of the beam, where $f(y)$ is a conveniently defined but arbitrary function introduced by Timoshenko.

Specifically, for the beam of circular cross section (radius r), a suitable form for $f(y)$ is

$$f(y) = \frac{P}{2I}(r^2 - y^2)$$

with this, the above equation for ψ becomes

$$\nabla^2 \psi = \frac{1+2\nu}{1+\nu}\frac{Py}{I}$$

with the solution

$$\psi = -\frac{1+2\nu}{8(1+\nu)}\frac{P}{I}(x^2+y^2-r^2)y$$

From this, the stress components are found to be

$$\sigma_{xz} = \frac{\partial\psi}{\partial y} - \frac{Px^2}{2I} + f(y) = \frac{(3+2\nu)}{8(1+\nu)}\frac{P}{I}(r^2 - x^2 - \frac{1-2\nu}{3+2\nu}y^2)$$

$$\sigma_{yz} = -\frac{\partial\psi}{\partial x} = -\frac{(1+2\nu)}{4(1+\nu)}\frac{Pxy}{I}$$

and, from elementary beam theory,

$$\sigma_{zz} = -\frac{P(L-z)x}{I}$$

13.3 Pressurized Spherical Shell

For instance, for a hollow sphere (inner radius a, outer radius b) subjected to inner pressure p and outer pressure q. Consider the potential function

$$\phi = \frac{C}{R} + DR^2$$

where C and D are constants and $R^2 = x^2 + y^2 + z^2$. This potential function can be used to determine the stress at $(R, 0, 0)$. The values of the constants C and D are determined by incorporating the boundary conditions to yield

$$\sigma_{RR} = -p\frac{(b/R)^3 - 1}{(b/a)^3 - 1} - q\frac{1-(a/R)^3}{1-(a/b)^3}$$

and

$$\sigma_{\theta\theta} = \frac{p}{2}(\frac{(b/R)^3 + 2}{(b/a)^3 - 1}) - \frac{q}{2}(\frac{(a/R)^3 + 2}{1-(a/b)^3})$$

13.4 Pressurized Cylindrical Tube

Consider a long cylindrical tube subjected to uniform pressures at its inner and outer surfaces ($-p$ at $r = a$ and $-q$ at $r = b$). Axial symmetry can be assumed and body forces may be neglected. The equilibrium equation becomes

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

Moreover, Hooke's law reduce to

$$\epsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = \frac{du}{dr}$$

and

$$\epsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) = \frac{u}{r}$$

where u is the radial displacement.

It can be shown that an appropriate form for the stress function that yields physically meaningful, single-valued displacement functions in this case is

$$\Phi = A\log r + Cr^2 + D$$

Taking derivatives of this and substituting the stated boundary conditions yields expressions for the unknown constants A and C and the stress field is given by

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\Phi}{\partial r} = -p\frac{(b/r)^2 - 1}{(b/a)^2 - 1} - q\frac{1 - (a/r)^2}{1 - (a/b)^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2} = p\frac{(b/r)^2 + 1}{(b/a)^2 - 1} - q\frac{1 + (a/r)^2}{1 - (a/b)^2}$$

And the displacement is given by

$$u = \frac{1-\nu}{E}\frac{(a^2p - b^2q)r}{b^2 - a^2} + \frac{1+\nu}{E}\frac{(p-q)a^2b^2}{(b^2 - a^2)r}$$

In this particular case, the problem can also be solved without involving the stress function by solving the equidimensional differential equation that results from substituting Hooke's law and the strain-displacement relationships into the equilibrium equation.

13.5 Bending of a Curved Bar

For the bending of a curved bar (inner curvature radius a, outer curvature radius b), by a force P acting towards the origin at one end while the other end is clamped, one has

$$\Phi = (d_1r^3 + \frac{c_1'}{r} + d_1'r\log r)\sin\theta$$

where

$$d_1 = \frac{P}{2N}$$

$$c_1' = -\frac{Pa^2b^2}{2N}$$

$$d_1' = -\frac{P}{N}(a^2 + b^2)$$

where $N = a^2 - b^2 + (a^2 + b^2)\log(b/a)$.

13.6 Rotating Disks

Consider the problem of a solid disk of material with density ρ, radius b, of uniform thickness and rotating about its center with angular velocity ω. If the disk thickness is small compared to its radius, both, radial and tangential stresses can be regarded approximately constant though the thickness Moreover, because of the symmetry, the stress components can be regarded as functions of r only and the shear stresses vanish. The equilibrium equations reduce to a single one, namely

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + X_r = \frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0$$

where $X_r = \rho\omega^2 r$.

It can be shown that a stress function F defined by the relationships

$$F = r\sigma_r$$

and

$$\frac{dF}{dr} = \sigma_\theta - \rho\omega^2 r^2$$

produces stresses satisfying the equilibrium equation.

The strain tensor components in this case are

$$\epsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\epsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

where u is the radial displacement and Hooke's law has been used.

By eliminating u from the above equations and using Hooke's law, the following differential equation for F is obtained

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - F + (3+\nu)\rho\omega^2 r^3 = 0$$

The general solution is of the form

$$F = Cr + \frac{C_1}{r} - \frac{3+\nu}{8}\rho\omega^2 r^3$$

Particular solutions to specific problems can now be obtained by introducing boundary conditions. For instance, for a solid disk, the stresses are finite at the center and σ_r vanishes at its outer radius $r = b$. The resulting stresses are then

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(b^2 - r^2)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 b^2 - \frac{1+3\nu}{8}\rho\omega^2 r^2$$

Alternatively, for a disk with a hole at the center (radius a), since $\sigma_r = 0$ at both $r = a$ and $r = b$, the stresses become

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(b^2 + a^2 - \frac{a^2 b^2}{r^2} - r^2)$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2(b^2 + a^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2)$$

Note that as the size of the hole approaches zero, the maximum tangential stress does not converge to the value obtained for the solid disk. This is the effect of stress concentration associated with the presence of the hole.

13.7 Plate with a Hole under Uniaxial Tension

Consider a large plate with a circular hole (radius a) in the middle and under uniform uniaxial tension S along the $x-$direction.

Far away from the hole (for radii $b >> a$), the stresses are given by

$$(\sigma_r)|_{r=b} = \frac{S}{2}(1 + \cos(2\theta))$$

and

$$\tau_{r\theta} = -\frac{S}{2}\sin(2\theta)$$

where θ is the angle between the position vector and the positive $x-$axis measured clockwise.

It can be shown that the above stresses can be obtained from the stress function

$$\phi = f(r)\cos(2\theta)$$

by using the equations

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

and

$$\tau_{r\theta} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \phi}{\partial \theta})$$

By requiring the stress function to satisfy

$$(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2})=0$$

the compatibility condition is automatically satisfied. Introducing the assumed relationship for ϕ yields

$$(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{4}{r^2})(\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{4f}{r^2})=0$$

This can be readily solved yielding the following general solution for the stress function

$$\phi=(AR^2+Br^4+\frac{C}{r^2}+D)\cos(2\theta)$$

Finally, introducing the far-field stress conditions (at $r=b$) and the condition of zero external forces at the hole boundary yields

$$\sigma_r=\frac{S}{2}(1-(\frac{a}{r})^2)+\frac{S}{2}(1+3(\frac{a}{r})^4-4(\frac{a}{r})^2)\cos(2\theta)$$

$$\sigma_\theta=\frac{S}{2}(1+(\frac{a}{r})^2)-\frac{S}{2}(1+3(\frac{a}{r})^4)\cos(2\theta)$$

$$\tau_{r\theta}=-\frac{S}{2}(1-3(\frac{a}{r})^4+2(\frac{a}{r})^2)\sin(2\theta)$$

13.8 Thermal Stresses in a Thin Plate

Consider an infinitely long plate of very small thickness and width $2c$. Let the long direction be aligned with the x axis and the width with y. Assume that $T_0=0$ and that the temperature in the plate is only a function of y, (i.e. $\theta=T(y)$). What would be the thermoelastic states of strain and stress resulting from this temperature field?

The answer is obtained using the principle of superposition. First, one must determine the amount of compressive stress that would have to be applied to keep the plate from straining altogether in the longitudinal (x) direction.

From the above, the required stress would be

$$\sigma'_x=-\alpha ET(y)$$

Since one is interested in the thermal stress in an expanding plate, to the above stress one must superimpose the stress generated in the plate when a uniformly distributed tensile force of magnitude

$$\frac{1}{2c}\int_{y=-c}^{y=+c}\alpha ET(y)dy$$

is applied at the $x \to \pm\infty$ boundaries.

Therefore, the actual **thermal stress** in the plate is

$$\sigma_x = \frac{1}{2c}\int_{y=-c}^{y=+c} \alpha E T(y) dy - \alpha E T(y)$$

Assume now that $T(y)$ is quadratic in y,

$$T(y) = T_{y=0}(1 - \frac{y^2}{c^2})$$

I.e. the center of the plate is at temperature $T_{y=0}$ while the edges $y = \pm c$ are at 0. Substituting this into the expression for σ_x gives

$$\sigma_x = \frac{2}{3}\alpha E T_{y=0} - \alpha E T_{y=0}(1 - \frac{y^2}{c^2})$$

Clearly, the stress is quadratic in y. The maximum compressive stress is at $y = 0$ and it is equal to $\sigma_{x,y=0} = -\frac{1}{3}\alpha E T_{y=0}$, while the maximum tensile stress is at $y = \pm c$ and it is $\sigma_{x,y=\pm c} = \frac{2}{3}\alpha E T_{y=0}$. The stress is zero at $y = \pm c/\sqrt{3}$.

13.9 Thermal Stress in Disks and Cylinders

Consider a thin disk (radius b) with a hole of radius a at the center. Assume the temperature in the disk $\theta = T(r)$ is only a function of the radial position r measured from the center of the hole.

If plane stress conditions are assumed, mechanical equilibrium requires

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\phi}{r} = 0$$

where r and ϕ are the radial and azimuthal directions, respectively.

The strain-displacement relations are

$$\epsilon_r = \frac{du}{dr}$$
$$\epsilon_\phi = \frac{u}{r}$$

where u is the radial displacement.

Finally, for linear thermoelastic material the stress-strain relations are

$$\sigma_r = \frac{E}{1-\nu^2}[(\epsilon_r + \nu\epsilon_\phi) - (1+\nu)\alpha T]$$

$$\sigma_\phi = \frac{E}{1-\nu^2}[(\epsilon_\phi + \nu\epsilon_r) - (1+\nu)\alpha T]$$

Combination of the strain-displacement relations with the above and substitution into the mechanical equilibrium equation yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = (1+\nu)\alpha\frac{dT}{dr}$$

with the general solution

$$u = (1+\nu)\alpha\frac{1}{r}\int_a^r Trdr + C_1\frac{r}{2} + C_2\frac{1}{r}$$

where C_1, C_2 are constants. The associated stresses are

$$\sigma_r = -\frac{\alpha E}{r^2}\int_a^r Trdr + \frac{E}{1-\nu}\frac{C_1}{2} - \frac{E}{1+\nu}\frac{C_2}{r^2}$$

$$\sigma_\phi = \frac{\alpha E}{r^2}\int_a^r Trdr - \alpha ET + \frac{E}{1-\nu}\frac{C_1}{2} + \frac{E}{1+\nu}\frac{C_2}{r^2}$$

Since no radial stresses act at the inner and outer the outer radius of the disk ($\sigma_r(a) = \sigma_r(b) = 0$),

$$C_1 = \frac{2(1-\nu)\alpha}{b^2-a^2}\int_a^b Trdr$$
$$C_2 = \frac{(1+\nu)\alpha a^2}{b^2-a^2}\int_a^b Trdr$$

and the axial strain is

$$\epsilon_z = (1+\nu)\alpha T - \frac{2\nu\alpha}{b^2-a^2}\int_a^b Trdr$$

If plane strain conditions are assumed instead (a good approximation in the case of a tall hollow cylinder with its bases restrained from movement along the axial direction), the corresponding results are, for the displacement

$$u = \frac{\alpha}{r}\frac{1+\nu}{1-\nu}[\int_a^r Trdr + \frac{(1-2\nu)r^2+a^2}{b^2-a^2}\int_a^b Trdr]$$

and for the associated stresses are

$$\sigma_r = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-\int_a^r Trdr + \frac{r^2-a^2}{b^2-a^2}\int_a^b Trdr]$$

$$\sigma_\phi = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-Tr^2 + \int_a^r Trdr + \frac{r^2+a^2}{b^2-a^2}\int_a^b Trdr]$$

and

$$\sigma_z = \alpha E \frac{1}{1-\nu}[-T + \frac{2\nu}{b^2 - a^2}\int_a^b Trdr]$$

The solution to the case of a tall hollow tube unrestrained from movement in the axial direction is given by

$$u = \frac{\alpha}{r}\frac{1}{1-\nu}[(1+\nu)\int_a^r Trdr + \frac{(1-3\nu)r^2 + (1+\nu)a^2}{b^2 - a^2}\int_a^b Trdr]$$

the associated stresses are

$$\sigma_r = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-\int_a^r Trdr + \frac{r^2 - a^2}{b^2 - a^2}\int_a^b Trdr]$$

$$\sigma_\phi = \frac{\alpha E}{r^2}\frac{1}{1-\nu}[-Tr^2 + \int_a^r Trdr + \frac{r^2 + a^2}{b^2 - a^2}\int_a^b Trdr]$$

and

$$\sigma_z = \frac{\alpha E}{1-\nu}[-T + \frac{2}{b^2 - a^2}\int_a^b Trdr]$$

and the longitudinal strain is

$$\epsilon_z = \frac{2\alpha}{b^2 - a^2}\int_a^b Trdr$$

Specifically, for a thin disk with radial steady state temperature distribution

$$T(r) = T_b - (T_b - T_a)\frac{\ln(b/r)}{\ln(b/a)}$$

where $T_a = T(a), T_b = T(b)$. the stresses are

$$\sigma_r = \frac{1}{2}\alpha E(T_b - T_a)[\frac{1-(a/r)^2}{1-(a/b)^2} - \frac{\ln(r/a)}{\ln(b/a)}]$$

$$\sigma_\phi = \frac{1}{2}\alpha E(T_b - T_a)[\frac{1+(a/r)^2}{1-(a/b)^2} - \frac{1+\ln(r/a)}{\ln(b/a)}]$$

with $\sigma_z = 0$. The corresponding stresses for the long hollow cylinder are obtained dividing the above by $1-\nu$. but in this case with

$$\sigma_z = \frac{\alpha E(T_b - T_a)}{2(1-\nu)}[\frac{2}{1-(a/b)^2} - \frac{1+2\ln(r/a)}{\ln(b/a)}]$$

13.10 Thermal Stresses in Quenching

Consider the case of quenching a long free cylinder, initially at a uniform temperature $T(r) = T_0$ by maintaining its surface temperature at zero ($T(r = b) = 0$). The solution of the homogeneous linear transient 1D heat conduction problem is (see for example "Conduction of Heat in Solids", 2nd ed, by Carslaw and Jaeger, Clarendon, Oxford, 1959, p. 199):

$$T(r) = T_0 \sum_{n=1}^{\infty} \frac{2}{\beta_n J_1(\beta_n)} J_0(\beta_n \frac{r}{b}) e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

where κ is the thermal diffusivity, J_0 and J_1 are the Bessel functions of first kind, of orders zero and one, respectively and β_n are the eigenvalues of the problem, which are the roots of

$$J_0(\beta_n) = 0$$

Substituting the expression for $T(r)$ into the stress equations one obtains

$$\sigma_r(r) = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} [\frac{1}{\beta_n^2} - \frac{1}{\beta_n^2} \frac{b}{r} \frac{J_1(\beta_n(r/b))}{J_1(\beta_n)}] e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

$$\sigma_\phi(r) = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} [\frac{1}{\beta_n^2} + \frac{1}{\beta_n^2} \frac{b}{r} \frac{J_1(\beta_n(r/b))}{J_1(\beta_n)} - \frac{J_0(\beta_n(r/b))}{\beta_n J_1(\beta_n)}] e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

and

$$\sigma_z(r) = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} [\frac{2}{\beta_n^2} - \frac{J_0(\beta_n(r/b))}{\beta_n J_1(\beta_n)}] e^{(-\kappa \frac{\beta_n^2}{b^2} t)}$$

If one is interested only in the maximum value of the stresses (which occur when $t \approx 0$ at the surface), the results are

$$\sigma_r(b) = 0$$

$$\sigma_\theta = \sigma_z = \frac{\alpha E T_0}{1-\nu}$$

I.e. the surface of the cylinder is under circumferential (hoop) and axial tensions of equal magnitudes. The (cold) surface layers of the cylinder want to contract but are prevented from doing so by the (still hot) core. If instead of quenching, a cold cylinder is heated, the initial stress state at the surface is compressive.

13.11 Thermal Stresses in a Sphere

Consider a sphere of radius b in which the temperature is only function of r. The differential mechanical equilibrium equation is

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_t) = 0$$

where σ_r and σ_t are, respectively, the radial and tangential stress.

The stress-strain relations are:

$$\epsilon_r = \alpha T + \frac{1}{E}(\sigma_r - 2\nu\sigma_t)$$

and

$$\epsilon_t = \alpha T + \frac{1}{E}[\sigma_t - \nu(\sigma_r + \sigma_t)]$$

Finally, the displacement-strain relationships are

$$\epsilon_r = \frac{du}{dr}$$

and

$$\epsilon_t = \frac{u}{r}$$

The solution in this case is

$$u(r) = \frac{1+\nu}{1-\nu}\alpha\frac{1}{r^2}\int_0^r Tr'^2 dr'$$

$$\sigma_r(r) = \frac{2\alpha E}{1-\nu}[\frac{1}{b^3}\int_0^b Tr^2 dr - \frac{1}{r^3}\int_0^r Tr'^2 dr']$$

and

$$\sigma_t(r) = \frac{\alpha E}{1-\nu}[\frac{2}{b^3}\int_0^b Tr^2 dr + \frac{1}{r^3}\int_0^r Tr'^2 dr' - T]$$

If $T(r)$ is known, stresses are readily computed.

For instance, if a cold solid sphere, initially at T_0, is heated by maintaining its surface at temperature T_1, the maximum compressive stress (occurring at the surface at the very beginning of the process) is

$$\sigma_r(b) = \sigma_t(b) = -\frac{\alpha E(T_1 - T_0)}{1-\nu}$$

13.12 The Finite Element Method for a One-Dimensional System

Consider an elastic bar loaded by a distributed internal body force $f(x)$ constrained at the end $x = 0$, attached to a rigid wall by a spring of spring constant k_l at $x = L$ and under imposed axial tractions $T_{ds} = cu(x)$ due to a distributed spring acting along its length. The goal is to determine the longitudinal displacement of the bar at equilibrium. If the cross sectional area of the bar is $A(x)$ the condition of mechanical equilibrium is obtained by performing a force balance on a small element Δx of the bar and then taking the limit as $|Deltax \rightarrow 0$. The result is

$$-\frac{d}{dx}(AE\frac{du}{dx}) + cu = f$$

The principle of virtual work is obtained by multiplying the equilibrium equation by a reasonable function $v(x)$ and then integrating from $x = 0$ to $x = L$ while using integration by parts the result is, after minor rearrangement

$$\int_0^L (AE\frac{du}{dx}\frac{dv}{dx} + cuv)dx = \int_0^L fvdx + (Fv)_{x=L} - (Fv)|_{x=0}$$

The function v is called the virtual displacement and dv/dx is the virtual strain. Introducing the notation

$$a(u,v) = \int_0^L (AE\frac{du}{dx}\frac{dv}{dx} + cuv)dx$$

and

$$(f,v) = \int_0^L fvdx + (Fv)|_{x=L} - (Fv)|_{x=0}$$

the principle of virtual work becomes

$$a(u,v) = (f,v)$$

Using the above notation one defines the elastic strain energy U as

$$U = \frac{1}{2}a(u,u)$$

As in the general formulation of the finite element methodology, the function $\mathbf{u}$ that satisfyies the differential mechanical equilibrium equations subject to given boundary conditions is the same that solves the variational problem expressed by the principle of virtual work and it is also the same that minimizes the energy functional.

14 Exercises

14.1

Consider a point inside an isotropic linear elastic solid where the stress tensor is given by

$$\sigma_{ij} = \begin{bmatrix} 2 \times 10^6 & -3 \times 10^6 & 1 \times 10^6 \\ -3 \times 10^6 & 4 \times 10^6 & 5 \times 10^6 \\ 1 \times 10^6 & 5 \times 10^6 & -1 \times 10^6 \end{bmatrix}$$

in Pascals. Use Hooke's law and determine the corresponding strain tensor components.

14.2

Show that the biharmonic equation for the Airy stress function Φ corresponding to two-dimensional, linear elastic systems in polar coordinates, reduces to the following form for the axially symetric case

$$\frac{d^4\Phi}{dr^4} + \frac{2}{r}\frac{d^3\Phi}{dr^3} - \frac{1}{r^2}\frac{d^2\Phi}{dr^2} + \frac{1}{r^3}\frac{d\Phi}{dr} = 0$$

Obtain the solution $\Phi(r)$ to the above equation. Then use the definitional relationships for the stress function

$$\sigma_{rr} = \frac{1}{r}\frac{d\Phi}{dr}$$

and

$$\sigma_{\theta\theta} = \frac{d^2\Phi}{dr^2}$$

to obtain expressions for the radial and azimuthal (hoop) stresses inside the wall of a long cylindrical pipe (inner radius a, outer radius b) pressurized internally to pressure p with the outside pressure maintained at zero. Make plots of the dimensionless stresses σ_{rr}/p and $\sigma_{\theta\theta}/p$ versus the dimensionless thickness r/b (for the range $a/b \leq r/b \leq 1$).

14.3

A cantilever beam has length $L = 1$, breadth $b = 1$ and height $h = 0.1$. The elastic modulus is $E = 10^{11}$ and Poisson's ratio is $\nu = 0.3$. At the lower edge of the free end, a downward load $F = 10^5$ N is applied. Use the Ansys finite element software with linear elements to determine approximate solutions to this problem. Compare your results against those obtained from elementary beam theory.

14.4

A simply supported beam has length $L = 1$, breadth $b = 1$ and height $h = 0.1$. The elastic modulus is $E = 10^{11}$ and Poisson's ratio is $\nu = 0.3$. In the middle of the upper edge of the beam, a downward point force $F = 10^5$ N is applied. Use the Ansys finite element software with linear elements to determine approximate solutions to this problem. Compare your results against those obtained from elementary beam theory.

14.5

The temperature inside a thin solid circular disk (radius b) is symmetrical about its center and has the form

$$T(r) = T_0(1 - \frac{r}{b})$$

where T_0 is a constant value. Hooke's law taking into account the thermal expansion effect gives the radial and azimuthal strain components

$$\epsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + \alpha T$$

$$\epsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + \alpha T$$

where u is the radial displacement and α the thermal expansion coefficient.

Solve the above for the stresses then substitute into the equilibrium equation appropriate for this case and obtain a differential equation for the radial displacement. Finally, solve the resulting differential equation and obtain expressions for $u(r), \sigma_r(r)$ and $\sigma_\theta(r)$.