Chapter 5
Plasticity

1 Introduction

One key goal of crystal plasticity studies is the mechanistic description of the process of strain or work hardening resulting from plastic deformation. It is a common observation that when metal forming operations are carried out at relatively low temperatures, strain hardening occurs whereas if the processing temperature is sufficiently high, the material does not harden significantly. These two forming regimes are known as cold working and hot working , respectively. It is now accepted that the hardening observed during cold working is the result of multiplication and interaction of crystalline defects, mainly dislocations. The density of dislocations (and the strength) usually increase rapidly at first and the rate of increase decreases with increasing strain. The frequently observed lack of hardening during hot working is the result from concurrent defect annihilation and rearrangement processes acting in tandem with the defect multiplication processes. Many commercial metal alloy products are available to designers and engineers in their work hardened state. This is the result of the previous manufacturing processes, often involving metal forming operations such as bending, stamping, rolling, forging or extrusion. This chapter presents an overview of metal plasticity. It includes a description of plastic deformation processes at the crystallographic and microstructural scales and also a presentation of concepts, theories and examples of continuum plasticity.

2 Crystal Plasticity

It is now widely agreed that atomic scale deformation processes in metals take place by a slippage process involving shear displacements. At moderate temperatures and strain rates, crystallographic slip predominates. This is characterized by shear displacement of a portion of the crystal over another that is a multiple of the interatomic spacing and leaves the lattice virtually unchanged. At low temperatures and/or high strain rates mechanical twinning dominates. In this case, the shear displacement is a fraction of the interatomic spacing, it involves shear between every successive plane and it results in mirror image lattice orientations of portions of the crystal on the sides of the twin. Furthermore, abrupt lattice reorientation is also involved. Since crystallographic slip is the dominant mechanism under most commonly encountered deformation conditions, this will be the focus of attention below.

2.1 Deformation Geometry

Most commercial metal and alloy products are polycrystalline materials consisting of assemblies of crystal grains. Inside each grain atoms are arranged in a relatively simple pattern (most commonly BCC, FCC or HCP) and they coexist with various amounts of crystalline defects such as vacancies, impurity and/or alloy atoms and dislocations. However, the defects are unable to significantly alter the three dimensional periodicity of the crystalline lattice.

When a polycrystalline material is subjected to sufficiently large external loads, individual grains deform plastically while constrained by the presence of the surrounding grains. Strain is transmitted from grain to grain while the grain assembly remains cohesive (i.e. no voids or cracks form inside the material). Eventually, if the load is sufficiently high, damage processes involving crack nucleation and growth may set it and eventually result in the ultimate failure of the component. However, for the loads commonly encountered in structural mechanics and in metal forming applications, the goal is to prevent failure and cracking.

Since the plastic deformation of a polycrystalline aggregate is ultimately the result of plastic deformation within individual crystal grains it is rather natural to investigate the plastic deformation behavior of metals by examining the deformation of single crystal specimens. Such specimens are carefully produced samples containing relatively small amounts of crystalline defects. Samples are fully characterized in terms of the crystallographic arrangement of their atoms using standard materials research laboratory techniques.

Observations performed on single crystal samples subjected to external loading indicate that plastic deformation often takes place in crystals by processes of slip on characteristic crystallographic planes and along specific crystal directions. The slip process itself involves the motion of dislocations on the slip plane and along the slip direction. As a dislocation completes its sweep through the slip plane an individual slip event is the result.

Typically, slip planes are characterized by possessing high atomic densities and their interplanar spacings are relatively large. Therefore, the slip plane and direction constitute preferred paths for crystallographic slip because of the relative weakness of interatomic bonding in the direction normal to the slip plane. The combination of crystallographic slip plane and direction is called a slip system and most crystals posses several potential slip systems. In order to understand the process of plastic deformation in crystals it is then necessary to be able to visualize and examine the crystallographic planes and directions associated with slip systems.

2.2 Representation of Crystallographic Features: The Stereographic Projection

A most useful tool for the representation of the crystallographic features (i.e. planes and directions) of single crystal samples is the stereographic projection. To visualize the projection imagine the single crystal sample is first placed at the origin of a system of coordinates oriented in a particular but arbitrary manner. Next, imagine the sample enclosed on one side by a transparent hemispherical surface centered at the origin and of a radius large enough so as to not touch the sample anywhere. Then, draw a normal line from each individual crystallographic plane up to the point of contact with the hemispherical surface and mark with a dot the point of intersection. Finally, to obtain the stereographic projection, imagine a light source located at the origin, shine light onto the collection of dots arranged on the transparent hemispherical surface and project them on a screen placed tangentially to it. The projection is effectively a two dimensional representation of the three dimensional features of the crystalline sample.

2.3 Schmid's Law

When a single crystal sample is pulled in tension the yield stress is very sensitive to the relative orientation of the sample with respect to the tension axis. The sensitivity is a natural consequence of the relative ease with which slip occurs along the various different slip systems in the material. Consider a specific slip system (crystallographic plane and direction) inside such sample. The relative orientation of the slip system with respect to the tension axis is specified by the angles ϕ and λ that the tension axis makes with the normal to the plane and with the direction, respectively. For a cylindrical sample of cross sectional area A subjected to load P, the resolved shear stress τ on the plane along the slip direction is simply given by

$$\tau = \frac{P \cos\lambda}{A/\cos\phi} = \sigma \cos\phi \cos\lambda = \sigma \sin\chi \cos\lambda$$

where $\sigma = P/A$ is the stress along the axis of the sample and χ is the supplementary angle to ϕ.

Schmid was the first to propose that a single crystal sample under load would begin to deform plastically (i.e. yield) along a particular slip system whenever the resolved shear stress acting on the plane along the slip direction reached a critical value, i.e.

$$\sigma_Y = \frac{1}{\sin\chi \cos\lambda}\tau_c = \frac{1}{M_s}\tau_c$$

where $M_s = \sin\chi \cos\lambda$ is called the Schmid factor and is a pure number with a maximum value of 0.5. The law is in very good agreement with experiment for HCP metals but no so good for FCC crystals. The reason being the much larger number of slip systems and the orientation dependent nature of M_s in the latter.

Another important finding concerning plastic deformation in single crystal samples is that the as strain increases the originally circular cross section of the sample becomes distorted as the slip direction gradually rotates towards the tensile axis. So for a sample of initial gage length L_0 that reaches length L_1 after some straining, the corresponding angles between the slip direction and the tension axis before and after deformation are λ_0 and λ_1. It can be shown that the above quantities are related as follows:

$$\frac{L_1}{L_0} = \frac{\sin \lambda_0}{\sin \lambda_1}$$

and that the associated shear strain γ is

$$\gamma = \frac{1}{\sin \chi_0}[\sqrt{(\frac{L_1}{L_0})^2 - \sin^2 \lambda_0} \cos \lambda_0]$$

where χ_0 is the original angle between the tension axis and the slip plane. Finally, the longitudinal strain ϵ is given in terms of the shear strain by $\epsilon = M_s \gamma$.

2.4 Slip Systems

The ease of slip depends strongly on the nature of the slip systems present in the sample. In HCP crystals is mainly restricted to the basal plane of the crystallographic unit cell. Therefore, a single slip system is operational over large ranges of strain. In BCC and FCC crystals, while only a single slip system may be active during the initial stages of deformation, the subsequent rotation can bring other slip systems into favorable orientations so that they become activated and also contribute to the strain.

A widely agreed qualitative picture of the deformation process in FCC single crystals that involves the shift from single slip to multiple slip is now available. An initially undeformed FCC single crystal will begin deforming plastically by slip when the critical resolved shear stress is reached on the most favorably oriented slip system. In a stress-strain curve this stage of the deformation process is characterized by a low rate of strain hardening and is called easy glide or stage 1 hardening. This stage is characterized by the appearance of long, thin and closely spaced slip lines that cover the crystal uniformly and by relatively low dislocation densities. As the load increases, slip begins to occur in other slip systems (conjugate slip), dislocations multiply and move on different systems and their interactions produce a much higher hardening rate where the stress is approximately linear with the strain. This is called stage 2 hardening and is characterized by increasing dislocation densities and the formation of peculiar (cell wall) dislocation substructures. Stage 2 is subsequently interrupted when higher stresses are able to make stuck dislocations circumvent obstacles by mechanisms such as cross slip involving stacking faults and thus reduce the rate of hardening. This last stage is known as stage 3 hardening and is characterized by an approximately parabolic dependency of stress on strain.

2.5 Polycrystals

Many commonly used engineering materials are used in polycrystalline form. Therefore, there has been much interest in determining how single crystal deformation behavior relates to the deformation of polycrystalline samples. Recall that in the single crystal case the tensile yield strength σ_Y was given in terms of the critical resolved shear stress τ_c by

$$\sigma_Y = \frac{1}{M_s}\tau_c = M\tau_c$$

where $M = 1/M_s$ is the reciprocal of the Schmid factor. Sachs first suggested that in applying the above to a polycrystal, one should simply average the values of M for all the individual crystal grains in the polycrystalline sample. Such analysis resulted in a an average value $\bar{M} = 2.238$ and thus $\sigma_Y = 2.238\tau_c$. Since this model assumes that a single slip system is operative in each crystal grain, and that the deformation of individual grains is unconstrained, the result proved to be only a lower bound for the strength in uniaxial tension.

An improved model is obtained if one removes the above assumptions. Specifically, in a polycrystal the straining of neighboring grains must be compatible in order to maintain cohesiveness.Von Mises first showed that a minimum number of active independent slip systems was required for strain compatibility under arbitrary shape changes of the aggregate and that this minimum number was five. Taylor then assumed that the actual five systems that became active were those requiring the least amount of work and established the following relationship

$$\sigma d\epsilon = \tau_c \sum_i d\gamma_i$$

or equivalently

$$\sigma = \tau_c \frac{\sum_i d\gamma_i}{d\epsilon} = M\tau_c$$

where σ and $d\epsilon$ are the macroscopic stress and strain increment in a uniaxial test and $\sum_i d\gamma_i$ is the collected strain increments of all the active slip systems. Taylor then calculated the average value of M for all orientations, $\bar{M} = 3.06$ and obtained the result

$$\sigma = 3.06\tau_c$$

The appropriateness of Taylor analysis was then confirmed by the good agreement shown with experiments on aluminum samples.

In contrast with single crystal samples polycrystals do not exhibit an easy glide stage of hardening since multiple slip systems are generally active. However, the observed hardening is still the result of dislocation motion, multiplication and interaction processes.

2.6 Work Hardening

Work hardening is strongly correlated with increasing dislocation densities in the deformed sample. Dislocations densities can vary from about 10^{12} dislocation lines per m^2 to over 10^{16} in severely cold worked samples. The following relationship has been frequently observed between flow stress of strained material, σ_s and its dislocation density ρ

$$\sigma_s = \sigma_Y + \alpha G b \sqrt{\rho}$$

where α is a numerical constant with a value between 0.3 and 0.6 and σ_Y is the stress required to move a dislocation in the absence of dislocation interactions (i.e. in a well annealed material). Various equivalent forms of this relationship is widely used as a basis for work hardening theories not only in metallic materials but also in ceramics.

Several theories have been proposed to quantitatively explain work hardening. In all cases the goal is to determine how dislocation density and distribution vary with strain. Because of the complexity of the processes involved, no definitive theory is yet available.

In Taylor's theory of work hardening dislocations interact with each other elastically and ultimately may become immobilized. The thus created sessile dislocations create back stresses that increase the stress required for deformation.

In Seeger's theory specific dislocation-based mechanisms are proposed for each of the three hardening stages in the single crystal stress-strain curve. The relatively unhindered dislocation motion on a single slip system that characterizes easy glide is followed by the onset of multiple slip and significant dislocation blockage in static obstacles such as Lomer-Cottrell locks. Stage 3 sets in when the long range internal stresses resulting from dislocation pileups are large enough to allow the dislocations to circumvent the obstacles by cross slip.

Kuhlmann-Wilsdorf has used the observed substructure evolution in cold formed metals to develop a work hardening theory. Substructures must be distinguished from microstructures. The latter are relatively coarse ($>$ 10 micrometers), they are reflected in the grain microstructure that is readily observed using metallographic light microscopy. The former consist of features in the submicron range and are usually only observable using electron microscopy methods.

At low stresses in well annealed materials, dislocation motion is relatively unhindered as there are few obstacles and mainly the line tension must be overcome. This situation is characteristic of stage 1 hardening. As strain increases, dislocations multiply and arrange into cellular structures with increasing strain. Continued dislocation motion requires that dislocations bow into cell interiors and on sweeping through they further increase their density while the cell size decreases. This situation characterizes stage 2 hardening. At even larger strains, dislocations in cell walls further rearrange themselves forming actual subgrain boundaries. By this process, subgrain interiors, containing relatively few dislocations become surrounded by dislocation tangles with high dislocation density. The result is stage 3 hardening.

2.7 Texturing

The word texture is used to describe the frequently observed anisotropy that results from pronounced plastic deformation of metallic samples. Texturing can be observed using conventional metallographic light microscopy and is characterized by the shape change of the crystalline grains in the microstructure. As a polycrystalline sample is subjected to increasing loads, individual crystalline grains deform and rotate while constrained by neighboring grains. Grains thus tend to reproduce the shape change induced in the whole sample. For instance, as a result of a heavy rolling operation, grains become severely elongated along the rolling direction. An immediate result of texturing is the development of anisotropic properties in an originally isotropic material. This anisotropy is of great practical importance in the processing of sheet stock such as in deep drawing where it is intimately connected with the phenomenon of earing.

3 Continuum Plasticity

In continuum plasticity, the complexity of crystallographic deformation mechanisms and evolving dislocation substructures is disregarded and the plastically deforming material is replaced by a homogeneous continuum. This is basically the same approach used in classical elasticity theory. Although plasticity mechanisms are the result of atomic defect processes a continuum phenomenological approach to the problem is often simpler and quite satisfactory in many applications.

As the material is loaded beyond its elastic limit, Hooke's law does not apply, the material yields, begins to flow and residual, permanent deformation results after unloading. This behavior is known as plasticity. Many materials deviate from linear elastic behavior at sufficiently high loads. Metals and alloys in particular often exhibit significant inelastic deformation before they fracture. Plastic response to unidirectional loading is readily determined using the tension test.

3.1 Uniaxial Tension Test

Recall that the uniaxial strain in the simple tension test of a bar of gage length L_0 can be expressed by the engineering strain as

$$\epsilon_e = \int_0^{\epsilon e} d\epsilon_e' = \int_{L_0}^{L} \frac{dL'}{L_0} = \frac{L - L_0}{L_0}$$

or the true strain

$$\epsilon = \int_0^{\epsilon} d\epsilon' = \int_{L_0}^{L} \frac{dL'}{L'} = \ln(\frac{L}{L_0})$$

Both strain measures are indistinguishable for small strains (say less than 0.02) but become quite different at large strains.

Likewise, the stress produced by the applied force F can be expressed in terms of the engineering stress

$$\sigma_e = \frac{F}{A_0}$$

or the true stress

$$\sigma = \frac{F}{A}$$

Again, both measures of stress are approximately equal at small strains but become significantly different for large strains.

The relationships between engineering and true quantities are

$$\sigma = (1 + \epsilon_e)\sigma_e$$

and

$$\epsilon = \ln(1 + \epsilon_e)$$

3.2 The Stress-Strain Curve

Most materials at room temperature behave elastically at small loadings so that the strain is proportional to the applied stress. At the elastic limit, Hooke's law does not apply anymore. Various forms of yielding behavior have been observed. Many commonly used iron based alloys exhibit a yielding plateau of fairly constant stress subsequent to a drop in the stress observed at the onset of yielding. At sufficiently large strains, the stress in these materials increases with strain in a nonlinear manner, this is called work hardening. On further straining, the material breaks. If the load is removed at any point beyond the initial yielding, there is some shrinkage of the bar $\epsilon^{(e)}$ but a residual amount of permanent plastic strain $\epsilon^{(p)}$ remains in the material.

Other materials do not exhibit the yielding plateau but simply display a nonlinear stress-strain relationship beyond the elastic limit. Finally, the strongest materials can withstand large loads while remaining elastic and they break quickly soon after the elastic limit is reached.

Meaningful physical parameters are often obtained from examination of simple tension test experiments. These include, the elastic modulus E, the linear elastic behavior proportionality limit , the yield stress $\sigma_0 = \sigma_Y$, the ultimate tensile stress σ_{UTS}, the uniform strain ϵ_u, the strain to failure ϵ_f, the resilience W_e and the work of fracture W_f.

3.3 Instability in Uniaxial Tension

While the state of stress during a tension test is rather simple at first, it becomes complex as soon as a localized necking instability develops in the sample. The condition for necking instability is that the rate of work hardening is matched by the rate of softening due to the reduced cross sectional area i.e.

$$Ad\sigma = -\sigma dA$$

Rearranging yields

$$\frac{d\sigma}{d\epsilon} = \sigma$$

i.e. necking takes place when the rate of strain hardening equals the stress.

Instability is readily recognized in experiments by the maximum in the engineering stress-strain curve, i.e.

$$\frac{d\sigma_e}{d\epsilon_e} = 0$$

As discussed in detail below, a commonly used expression for the approximate representation of the stress-strain behavior in plastically deforming material is of the form

$$\sigma = K\epsilon^n$$

From the above, one thus obtains the maximum strain at necking as

$$\epsilon_u = n$$

3.4 Related Phenomena

Several other important phenomena are often experimentally observed in simple tests. If a material is loaded past the elastic limit and exhibits strain hardening then, if the material is unloaded and subsequently loaded again, a new, higher elastic limit is obtained. The yield stress of the deformed material is larger than that of the originally undeformed material. Another common finding is that plastic deformation occurs under incompressibility conditions, i.e. no volume change. Further, when a material is first loaded in uniaxial tension past the yield point and then reverse loaded in compression, the yield point under compression is smaller than that obtained under the previous tensile load. This is known as Bauschinger effect. A related phenomenon is the development of anisotropy in an initially isotropic material that undergoes plastic deformation. Preferential texture is often the result of sustained plastic deformation. When plastic deformation takes place at high rates of strain the yield behavior is different to the one obtained at smaller strain rates; this is called strain rate sensitivity of plastic deformation. Temperature also affects significantly the inelastic deformation response. At sufficiently high temperatures materials undergo high temperature creep which is a form of time dependent plastic flow. Most materials consist of mixes of heterogeneous microstructural constituents, often with markedly different plastic deformation behavior. Microstructural effects can then play a key role in the plastic deformation response of the material. Finally, one is often interested in evaluating the plastic deformation response under multi axial loading conditions. Under these conditions, yielding must be specified by a function in stress space. Yet, one would like to be able to use the results of simple uniaxial loading experiments to produce appropriate constitutive relations for multi axial loading. Useful approximately biaxial loading experiments have been carried out using thin-walled tubes which are simultaneously loaded axially, by torsion and by internal pressure. Moreover, experiments carried out under large hydrostatic loading have shown than yielding is not affected by hydrostatic pressure.

4 Constitutive Models of Inelastic Behavior under Monotonic Uniaxial Loading

The elastic limit is the stress above which permanent deformation appears during a simple tension test. As a material is loaded beyond its elastic limit, Hooke's law does not apply, the material yields, begins to flow and residual, permanent deformation results after unloading. This behavior is known as plasticity. Although plasticity mechanisms are the result of atomic defect processes a continuum, phenomenological approach to the problem is often simpler and quite satisfactory in many applications. A characteristic feature frequently encountered during plastic deformation is that the material strain hardens. This phenomenon is called work hardening.

It is generally assumed that the total strain observed can be simply partitioned into an elastic strain and an inelastic or plastic strain, i.e.

$$\epsilon = \epsilon^{(e)} + \epsilon^{(p)}$$

This is the decoupling assumption. While elastic deformation is generally acknowledged to be the result of changes in the interatomic spacings, plastic deformation is the result of crystallographic slip processes.

Key issues that must be addressed by constitutive theory include the existence of a stress-strain threshold separating elastic and plastic behavior, the extent of plastic flow and the frequently observed strain hardening resulting from plastic deformation.

The threshold condition simply states that plastic flow will take place only when

$$|\sigma| \geq \sigma_Y$$

Concerning the extent of plastic flow, this is usually obtained by decoupling the constitutive equations for $\epsilon^{(e)}$ and $\epsilon^{(p)}$ as indicated above.

While Hooke's law describes well linear elastic behavior, the quantitative description of plastic deformation is more complicated and various alternative descriptions have been proposed. Three simple limiting forms of inelastic behavior are

- Rigid elastic, perfectly plastic. Here the material does not deform until the threshold condition is reached. Subsequently it flows at constant stress.
- Linear elastic , perfectly plastic. In this case the material behaves elastically for stresses below the threshold and it deforms plastically at constant stress after that.
- Linear elastic, linear plastic. The material is elastic until the threshold then deforms according to a linear stress-strain curve but with a slope much smaller than the one associated with the preceeding elastic behavior.

These can be used to represent inelastic behavior in a number of cases. Other formulae have been used to represent the results of tensile tests that do not fit any of the above behaviors.

The following expressions are used to represent the behavior of perfectly plastic solids:

$$\sigma < \sigma_Y \rightarrow \epsilon = \epsilon^{(e)} = \sigma/E$$

in the elastic regime, and

$$\sigma = \sigma_Y \rightarrow \epsilon = \epsilon^{(e)} + \epsilon^{(p)}$$

in the plastic regime. Note that the since the stress remains constant during plastic deformation in the case of a perfectly plastic solid, the plastic strain in the expression above has the same sign as σ but it is arbitrary.

In contrast, for hardening plastic solids the stress increases during plastic deformation and one has instead

$$\epsilon = \epsilon^{(e)} + \epsilon^{(p)}$$

where

$$\epsilon^{(e)} = \sigma/E$$

and

$$\epsilon^{(p)} =< \frac{|\sigma| - \sigma_Y}{K_Y} >^{M_Y} Sgn(\sigma)$$

Here, the operator $< x >$ produces x if $x > 0$ and zero otherwise; the operator $Sgn(x)$ is simply the sign of x. If the elastic contribution is neglected (i.e. rigid elastic solid), the above simplifies to

$$\epsilon^{(p)} = (\frac{|\sigma|}{K})^M Sgn(\sigma)$$

It is a simple matter to express the relationship between plastic strain and stress in the case of linear hardening, namely

$$\sigma_Y = \sigma_Y^0 + E^p \epsilon^{(p)}$$

where E^p is a plastic modulus. However, few materials exhibit linear hardening over extended ranges of strain.

A useful specific representation of more general hardening plastic solids is provided by the Ludwick-Hollomon equation (sometimes also called the Ramberg-Osgood equation),

$$\sigma = \sigma_0 + K(\epsilon^{(p)})^n = \sigma_Y^0 + K_Y(\epsilon^{(p)})^{1/M_Y}$$

where $\sigma_0 = \sigma_Y^0$ is the yield stress of the initial undeformed material, $K = K_Y$ and $n = 1/M_Y$ are material constants where n is called the work hardening exponent. Inverting the last expression yields the plastic strain as

$$\epsilon^{(p)} =< \frac{\sigma - \sigma_Y}{K_Y} >^{M_Y}$$

so that

$$\epsilon = \epsilon^{(e)} + \epsilon^{(p)} = \frac{\sigma}{E} + \alpha \frac{\sigma_R}{E} (\frac{\sigma}{\sigma_R})^{1/n}$$

where σ_R is a reference stress and α and m are dimensionless constants.

At large deformation, the contribution of the elastic strain to the total strain is negligible and the above reduces to a simple power law equation of the form

$$\sigma = K\epsilon^n$$

Other useful expressions that have been proposed include the Voce equation

$$\frac{\sigma_s - \sigma}{\sigma_s - \sigma_0} = \exp(-\frac{\epsilon}{\epsilon_c})$$

where σ_s, σ_0 and ϵ_c are empirical parameters; and the Johnson-Cook equation

$$\sigma = (\sigma_0 + K\epsilon^n)(1 + C\ln(\frac{\dot{\epsilon}}{\dot{\epsilon}_0}))[1 - (\frac{T - T_r}{T_m - T_r})^p]$$

where $\dot{\epsilon} = d\epsilon/dt$ and which has been shown to represent the effects of work hardening, strain rate and thermal softening.

Strain rate effects alone are often represented in terms of the strain rate sensitivity parameter m defined as

$$m = \frac{\partial \ln \sigma}{\partial \ln(\dot{\epsilon})} = \frac{\ln(\sigma_2/\sigma_1)}{\ln(\dot{\epsilon}_2/\dot{\epsilon}_1)}$$

where the subscripts 1 and 2 correspond to two different strain rates.

Cyclic loadings are often encountered in engineering structures. Under cyclic loading conditions involving stresses exceeding yielding a variety of hardening responses may be encountered. Cyclic loading tests may be performed under controlled strain or controlled stress conditions. Tests may also be symmetric or asymmetric. Materials may harden or soften under continued cycling but often a stabilized cycle sets in.

For symmetric cycling under controlled stress range $\Delta\sigma$ about the origin, the resulting plastic strain range $\Delta\epsilon^{(p)}$ associated with the stabilized cycle can be expressed by a relationship of the form

$$\Delta\epsilon^{(p)} = (\frac{\Delta\sigma}{K_c})^{M_c}$$

where K_c and M_c are empirically determined material parameters. Depending on the cyclic hardening response of the material, the cyclic hardening curve may be above or below its monotonic hardening counterpart.

5 Constitutive Models of Inelastic Behavior under Multi axial Loading

Multi axial loading conditions are encountered in many applications of structural mechanics. Yielding and hardening criteria obtained under uniaxial loading conditions must be extended to deal with multi axial loads.

The threshold condition for the case of solids that harden isotropically can be expressed as

$$f(J_2, J_3, \sigma_Y) = 0$$

where J_i are the invariants of the stress deviation tensor.

Concerning the flow law, effective stress and strain measures associated with multi axial loading conditions are introduced. These are defined as

$$\sigma_{eff} = \frac{\sqrt{2}}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} = (\frac{3}{2}\sigma'_{ij}\sigma'_{ij})^{1/2}$$

and

$$d\epsilon_{eff} = \frac{\sqrt{2}}{3}[(d\epsilon_1 - d\epsilon_2)^2 + (d\epsilon_2 - d\epsilon_3)^2 + (d\epsilon_3 - d\epsilon_1)^2]^{1/2} = [\frac{2}{3}(d\epsilon_1^2 + d\epsilon_2^2 + d\epsilon_3^2)]^{1/2}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses, σ'_{ij} are the components of the stress deviator tensor, $d\epsilon_1, d\epsilon_2, d\epsilon_3$ are the principal strain increments.

In terms of the above, the flow curve can be expressed, for instance, in terms of the power law equation as

$$\sigma_{eff} = K\epsilon_{eff}^n$$

Simple yet widely useful yielding criteria were first proposed by Tresca and von Mises. Tresca suggested that yielding under multi axial loading took place when the tangential shear stress reached its maximum value

$$f = Sup_{i \neq j}(|\sigma_i - \sigma_j|) - \sigma_Y = 0$$

However, the Tresca criterion in this form is difficult to apply. In the three dimensional space formed by the principal deviator stresses, the criterion is represented as a prism of hexagonal cross section with its axis equally inclined to the three coordinate axes.

Von Mises first proposed instead that yielding occurs when

$$f = \sigma_{eff} - \sigma_Y = \frac{1}{\sqrt{2}}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} - \sigma_Y = 0$$

In the three dimensional space formed by the principal deviator stresses, the criterion is represented as a prism of circular cross section (radius $R = \sqrt{\frac{2}{3}}\sigma_Y$), with its axis equally inclined to the three coordinate axes.

6 Plastic Solid obeying von Mises' Yield Criterion and Flow Rule

The macroscopic, continuum approach is generally based on the use of incremental strains and/or strain rates. Specifically, for a material loaded beyond the elastic limit the incremental strain deviation de'_{ij} is computed as the sum of the incremental plastic strain $de^{(p)}_{ij}$ and the incremental deviatoric elastic strain (the one that would be obtained if Hooke's law were still applicable) $de'^{(e)}_{ij}$, i.e.

$$de'_{ij} = de^{(p)}_{ij} + de'^{(e)}_{ij} = de^{(p)}_{ij} + \frac{d\sigma'_{ij}}{2G}$$

The above equation can be expressed in rate form by simply taking the time derivative and the resulting total plastic strain $e^{(p)}_{ij}$ is then given as

$$e^{(p)}_{ij} = e^{(p)}_{ij}(0) + \int_0^t \dot{e}^{(p)}_{ij} dt$$

Analysis of plastic deformation involves specification of the loading conditions required for yielding as well as expressions for the computation of the plastic strain resulting from plastic flow. von Mises proposed a useful yield criterion and its associated flow rule. His specification consists of the following:

- Mean stress and mean strain obey Hooke's law even after yielding. Therefore $e^{(p)}_{ii} = 0$ and $e^{(p)}_{ii} = e'^{(p)}_{ii}$.

- If $J_2 = \frac{1}{2}\sigma'_{ij}\sigma'_{ij} < k^2$ where k is a constant, the material remains elastic (i.e. $e_{ij}^{(p)} = 0$). The constant k directly related with the measured values of yield stress in simple uniaxial tension or shear experiments.

- Yielding only starts whenever $J_2 = k^2$ and once it starts, the plastic strain rate is simply assumed proportional to the stress deviation, i.e.

$$\dot{e}_{ij}^{(p)} = \frac{\sigma'_{ij}}{\mu}$$

  where $\mu > 0$ is a proportionality constant.

- If the amount of plastic flow is small, the deformation may be regarded as rate insensitive and the incremental strain is instead given by

$$de_{ij}^{(p)} = \lambda \sigma'_{ij}$$

  where the factor of proportionality $\lambda > 0$ can be a function of strain and stress and must not be confused with the Lame elasticity constant with the same name.

- Situations where $J_2 > k^2$ are impossible.

Because von Mises expresses the flow rule in rate form and since the total plastic strain must be computed by adding successive strain increments, his specification is known as a variant of an incremental theory of plasticity.

7 More General Yield Criteria and Yield Functions

The states of stress and strain at a point in a continuum are specified by the components of stress σ_{ij} and strain e_{ij}. To characterize plastic response, internal variables $\xi_1, \xi_2, ..., \xi_n$ are often used. These internal variables may include both physical variables (such as the population of structural defects) as well as phenomenological variables (such as the plastic strain $e^{(p)}$). In this representation, the function $f(\sigma_{ij}, T, \xi_i) = 0$ constitutes the yield surface. The material is elastic whenever $f < 0$ and deforms plastically whenever the yield surface is reached. Various distinct yielding criteria are obtained depending on 'the specific form of f adopted.

The continuum description of plastic deformation under general loading conditions requires specification of the initial yield surface, its evolution with the progress of deformation and the constitutive equations relating stress and strain. A useful first approximation consists of disregarding the rate sensitivity of the plastic deformation.

Experiments have shown that at low loads, recoverable elastic deformation prevails. As the proportionality limit is crossed, yielding soon follows. Yielding can then take place at approximately constant stress (as in many steels) or it may occur with strain hardening. A material unloaded from the strain hardened portion of the flow curve will recover only the elastic part of the strain and on subsequent loading will exhibit a yield stress higher than the original.

Consider a point in a plastic material where the stress is σ_{ij}, the strain is e_{ij} and the plastic component of the strain is $e_{ij}^{(p)}$. The yield function $f(\sigma_{ij}, T, \xi_i)$ in this situation is an expression that has the following properties

- The equation $f(\sigma_{ij}, T, \xi_k) = 0$ is a closed surface in the stress space σ_{ij} for any temperature T and combination of internal variables $\xi_k, k = 1, 2, .., n$.
- Both the plastic strain rate $de_{ij}^{(p)}/dt$ and all the internal variables are zero whenever the representative point gives $f(\sigma_{ij}, T, \xi_k) < 0$.
- The plastic strain rate can be nonzero when $f(\sigma_{ij}, T, \xi_k) = 0$.
- The condition $f(\sigma_{ij}, T, \xi_k) > 0$ is meaningless.

The internal variables ξ_k can be physical, such as the density of structural defects or phenomenological such as the total plastic strain and they characterize the strain hardening of the material resulting from plastic deformation. They are also called work hardening parameters and often depend on the detailed plastic deformation history as well as on the values of σ_{ij} and T.

As mentioned before, according to von Mises, the yield condition is defined in terms of the yield function

$$f(\sigma_{ij}) = J_2 - k^2$$

where $J_2 = \frac{1}{2}\sigma'_{ij}\sigma'_{ij}$ is the second invariant of the stress deviator tensor and k is a material constant whose value may depend on the strain history and specifically, under simple shear loading, is the yield stress in simple shear. Therefore, the von Mises yield criterion states that yielding will occur when

$$f(\sigma_{ij}) = J_2 - k^2 = 0$$

As mentioned before, the Tresca yield criterion is expressed by the function f in terms of the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ as follows

$$f(\sigma) = (\sigma_1 - \sigma_3) - 2k = 0$$

More generally, the Tresca yield criterion is expresses as

$$f(\sigma) = 4J_2^3 - 27J_3^2 - 36k^2J_2^2 + 96k^4J_2 - 64k^6 = 0$$

where J_2 and J_3 are the second and third invariants of the stress deviator tensor.

7.1 Experimental Results

To compare the above consider a thin walled pipe under uniaxial tension $\sigma > 0$ and shear stress τ. Yielding, according to the von Mises criterion will occur when

$$\sqrt{\sigma^2 + 3\tau^2} = \sqrt{3}k$$

and according to Tresca, when

$$\sqrt{\sigma^2 + 4\tau^2} = 2k$$

Under simple shear loading ($\sigma = 0; \tau = k$), $k = \sigma_Y/\sqrt{3}$ according to von Mises and $k = \sigma_Y/2$ according to Tresca.

Lode tested thin walled tubes (mean radius R, wall thickness t), under combined uniaxial tensile force F and internal pressure p, the resulting non-zero stresses were

$$\sigma_1 = \sigma_\theta = \frac{pR}{t}$$

and

$$\sigma_2 = \sigma_z = \frac{F}{2\pi R t}$$

Introducing the parameter $\eta = (2\sigma_2 - \sigma_1 - \sigma_3)/(\sigma_1 - \sigma_3)$ leads to produces the von Mises yield criterion as

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = \frac{2}{\sqrt{3 + \eta^2}}$$

and the Tresca criterion as

$$\frac{\sigma_1 - \sigma_3}{\sigma_Y} = 1$$

7.2 Loading and Unloading

Since $f = 0$ defines the yield surface, whenever $f = 0$ and $df/dt < 0$ leads to $f < 0$ subsequently. This is called unloading and it brings the representative stress point into an elastic state. By expressing the total time derivative of f in terms of the partial derivatives with respect to the independent variables, unloading involves $f = 0$ and

$$\frac{\partial f}{\partial \sigma_{ij}} \frac{d\sigma_{ij}}{dt} < 0$$

while loading requires $f = 0$ and

$$\frac{\partial f}{\partial \sigma_{ij}} \frac{d\sigma_{ij}}{dt} > 0$$

In the above, $\frac{\partial f}{\partial \sigma_{ij}}$ is directed outward normal to the surface $f = 0$.

7.3 Isotropic Theories of Yielding

An isotropic material does not exhibit orientation effects of deformation. If the yield function is an isotropic function of stress one has the isotropic stress theory of yielding. In this case the function f is of the forms

$$f(I_1, I_2, I_3)$$

or

$$f(\sigma_1, \sigma_2, \sigma_3)$$

where I_1, I_2, I_3 are the invariants of the stress tensor σ_{ij} and $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses.

Taking for instance $f = J_2 - k^2$, the surface $f = 0$ is a circular cylinder with its axis equally inclined to the three reference axes $(\sigma_1, \sigma_2, \sigma_3)$. The cross section of the cylinder is called the π-plane.

Since yielding is practically independent of hydrostatic pressure it is common to express the yield function in terms of the invariants of the stress tensor deviator, i.e $f(J_2, J_3)$. In von Mises case, simply $f = J_2 - const.$ and the yield surface is a circle on the π plane. Alternatively if $f(\sigma_1 - \sigma_3, \sigma_3 - \sigma_3)$, $f = 0$ becomes a polygon on the π-plane.

Other Yield Functions have been proposed to account for cases where the simple criteria above are not adequate. Some examples are

$$f(\sigma) = J_2^3 - 2.25 J_3^2 - k^2$$

which has been validated against some experimental data;

$$f(\sigma) = F(J_2) - m\sigma'_{ij} e_{ij}^{(p)} - k^2$$

where m is a constant and which can account for Bauschinger effect, and

$$f(H_{ijkl}\sigma'_{ij}\sigma'_{kl})$$

where H_{ijkl} is a fourth order material tensor and can account for yielding in anisotropic materials.

Special yield functions have also been introduced to account for the yielding behavior of pressure sensitive materials such as soils, rocks and porous materials.

7.4 Ideal Plasticity

Ideal plasticity is plastic deformation without hardening, i.e. for stress and strain increments $d\sigma_{ij}, de_{ij}$ necessarily

$$d\sigma_{ij} de_{ij}^{(p)} = 0$$

Moreover, for an the flow of an ideal plastic solid $\dot{e}_{ij}^{(p)} \neq 0$ if and only iff $f(\sigma_{ij}) = 0$ or equivalently $df = 0$. The incremental plastic strain for ideal plastic flow is then given by

$$de_{ij}^{(p)} = d\Lambda \frac{\partial f}{\partial \sigma_{ij}}$$

where $d\Lambda$ is a constant of proportionality. This is the rule of plastic flow in ideal plasticity. Finally, the general stress-strain increments relationship is of the form

$$d\sigma_{ij} = D_{ijkl} de_{kl}$$

where D_{ijkl} is a fourth order tensor.

7.5 Work Hardening of Continua

If during plastic deformation the stress is a monotonically increasing function of strain one has work hardening. Consider a point in a deforming material where the stress and strain tensor components are σ_{ij} and e_{ij} and assume external loading perturbs this state by $d\sigma_{ij}$ and de_{ij}. If the perturbation of stress is now removed, only the elastic portion of the perturbed strain $de_{ij}^{(e)}$ is recovered while the plastic portion of the perturbed strain $de_{ij}^{(p)}$ is not recovered. Therefore, a material is said to work harden whenever

$$d\sigma_{ij} de_{ij} > 0$$

upon loading and

$$d\sigma_{ij} de_{ij}^{(p)} \geq 0$$

on completion of the loading-unloading cycle. The last equation was expressed by Drucker for any generic loading produced by an external agency that brings the representative point from $\bar{\sigma}_{ij}$ to σ_{ij} leading to

$$(\sigma_{ij} - \bar{\sigma}_{ij}) \frac{de^{(p)}}{dt} \geq 0$$

So, if the external agency does positive work on a material during an elasto-plastic stress cycle, the material exhibits hardening. This equation is known as the principle of maximum plastic dissipation.

7.6 Flow Rule

Von Mises first suggested that the plastic strain rate could be expressed in terms of a plastic potential function $h(\sigma_{ij})$ as

$$\dot{e}_{ij}^{(p)} = \lambda \frac{\partial h}{\partial \sigma_{ij}}$$

where the potential could be the yield function (i.e. $h = f$) as in the case of the ideal plastic body or the plastic deformation of metals, or not, as in the case of porous materials. In the first case one has associated plasticity and in the second, non-associated plasticity.

Drucker proposed a precise definition of the term work hardening that requires the fulfillment of the following three conditions:

- Initial yield surface and subsequent loading surfaces must be convex.
- The plastic strain increment is normal to the loading surface.
- The rate of change of strain is proportional to the rate of change of stress.

From the above, one can derive the following generic constitutive equation relating the incremental stresses and strains for plastic deformation of work hardening materials

$$de_{ij} = (C_{ijkl} + \frac{h_{ij}n_{kl}}{K^p})d\sigma_{kl}$$

where C_{ijkl} is the elastic flexibility tensor and

$$K^p = -(\frac{\partial f}{\partial e_{rs}^{(p)}} + \frac{\partial f}{\partial \kappa}\frac{\partial \kappa}{\partial e_{rs}^{(p)}})h_{rs}(\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}})^{-1/2}$$

is the plastic modulus where $e_{ij}^{(p)}$ and κ are internal variables and

$$h_{ij} = \frac{\partial h}{\partial \sigma_{ij}}(\frac{\partial h}{\partial \sigma_{kl}}\frac{\partial h}{\partial \sigma_{kl}})^{-1/2}$$

In terms of the plastic stress and strain rates the flow rule is

$$\dot{e}_{ij}^{(p)} = \frac{h_{ij}n_{kl}\dot{\sigma}_{kl}}{K^p}$$

where

$$n_{ij} = \frac{\partial f}{\partial \sigma_{ij}}(\frac{\partial f}{\partial \sigma_{kl}}\frac{\partial f}{\partial \sigma_{kl}})^{-1/2}$$

is the unit normal to the yield surface at the loading point.

Hardening rules are specifications of how the internal variables affect the shape and size of the loading surface $f(\sigma_{ij}, T, \xi_k)$. Recall that for plastic deformation independent of hydrostatic pressure, yield surfaces in the space of principal stresses are cylinders equally inclined to the three axes and the plane of the cross section of the cylinder is the π plane. For work hardening materials the size and/or shape of the cylinder cross section can change during plastic deformation. If the yield surface simply expands equally in all directions as a result of plastic deformation then one has isotropic hardening. Alternatively, if the yield surface does not change in size but rather shifts in stress space then one has kinematic hardening.

Various combinations of a yield criterion and the flow rule are thus possible as follows:

- Isotropic yield criterion -Isotropic hardening;
- Isotropic yield criterion -Kinematic hardening;
- Anisotropic yield criterion-Isotropic Hardening;
- Anisotropic yield criterion-Kinematic Hardening;
- Isotropic yield criterion-Anisotropic, Non-kinematic Hardening;
- Anisotropic yield criterion-Anisotropic Hardening.

7.7 Isotropic Hardening

In isotropic hardening the material remains isotropic during deformation and the yield surfaces that develop during deformation can then be represented in terms of a single scalar internal variable κ as follows

$$f = f^*(J_2, J_3) - \kappa = 0$$

The internal variable κ is a monotonically increasing function of the effective plastic strain but does not depend on the strain path. The effective plastic strain is defined as

$$e_{eff}^{(p)} = \int de_{eff}^{(p)} = \int \sqrt{\frac{2}{3} de_{ij}^{(p)} de_{ij}^{(p)}}$$

The above definition is used so that the effective strain is equal to the total uniaxial plastic strain in simple uniaxial loading.

Combining the generic constitutive equation for work hardening material with the above leads to

$$\dot{e}_{eff}^{(p)} = \sqrt{\frac{2}{3}} \frac{n_{kl} \dot{\sigma}_{kl}}{K^p}$$

where

$$K^p = \sqrt{\frac{2}{3}} \frac{\partial \kappa}{\partial e_{eff}^{(p)}} \left(\frac{\partial f}{\partial \sigma_{kl}} \frac{\partial f}{\partial \sigma_{kl}} \right)^{-1/2}$$

So, once K^p is determined, $\dot{e}_{eff}^{(p)}$ and $\dot{e}_{ij}^{(p)}$ can be calculated.

A commonly used flow law for the elastoplastic regime with isotropic hardening is described by the Prandtl-Reuss equation. To obtain the flow law it is assumed that

- Flow is independent of hydrostatic stress;

- The loading function depends only on the deviatoric stress and the internal variable;
- The material is isotropic and it also hardens isotropically;
- Associated plasticity and normality of the plastic strain increments;
- von Mises yield criterion applies (i.e. $f = \sigma_{eff} - \kappa - \sigma_Y = 0$

The Prandtl-Reuss equations are

$$de_{ij}^{(e)} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk}\delta_{ij}$$

$$de_{ij}^{(p)} = \frac{3}{2} H(f) g'(\sigma_{eff}) \frac{< d\sigma_{eff} >}{\sigma_{eff}} \sigma'_{ij} =$$
$$= \frac{3}{2}\frac{M_Y}{K_Y} < \frac{\sigma_{eff} - \sigma_Y}{K_Y} >^{M_Y - 1} \frac{< d\sigma_{eff} >}{\sigma_{eff}} \sigma'_{ij}$$

where the values of the parameters g' and κ are determined from the results of simple uniaxial loading experiments as $g' = (\frac{d\sigma}{d\epsilon})^{-1}$ and $\kappa(p) = \sigma - \sigma_Y = K_Y p^{1/M_Y}$. In the case of perfect plasticity there is no strain hardening, $\kappa = 0$ and the plastic strains are indeterminate, i.e.

$$de_{ij}^{(p)} = \frac{3}{2} d\lambda \frac{\sigma'_{ij}}{\sigma_{eff}}$$

where $d\lambda > 0$ is arbitrary.

An equivalent representation of the Prandtl-Reuss equation is

$$de_{ij} = de_{ij}^{(e)} + de_{ij}^{(p)}$$

with

$$de_{ij}^{(e)} = \frac{1+\nu}{E} d\sigma'_{ij} + \frac{1-2\nu}{E}\frac{d\sigma_{kk}}{3}\delta_{ij}$$

and

$$de_{ij}^{(p)} = \frac{3}{2}\frac{de_{eff}}{\sigma_{eff}}\sigma'_{ij}$$

If the elastic contribution to the strain is neglected in the above, one obtains another set of commonly used stress-strain relations; the Levy-Mises Equation.

7.8 Kinematic Hardening

Kinematic hardening is the translation of the loading surface on the π plane during plastic deformation without change of its shape. A useful pictorial representation of kinematic hardening was first provided by Prager. In Prager's picture, the yield surface is represented by a hollow frame resting on a frictionless flat surface. The representative loading point is represented by a movable frictionless pin located inside the hollow space of the frame. On loading, the pin moves towards the edge of the hollow space and it eventually engages the frame. On further loading the pin simply pushes the frame in the direction of its own motion. Finally, unloading is represented by the pin disengaging and moving away from the frame. The Bauschinger effect and anisotropic hardening are also readily visualized using Prager's mechanical model.

While isotropic hardening is represented in terms of a single scalar hardening variable, kinematic hardening must be represented by a tensorial hardening variable α_{ij}, often called the back stress and representing the translation of the center of the initial yield surface during loading. . If the initial yield surface of the material is given by $f(\sigma_{ij}) = 0$, the surface obtained following some kinematic hardening is

$$f(\sigma_{ij} - \alpha_{ij}) = 0$$

The key assumption in Prager's linear kinematic hardening model is that the yield surface moves in the direction of the plastic strain rate, i.e.

$$\dot{\alpha}_{ij} = c\dot{e}_{ij}^{(p)}$$

where c is a constant, equal to K^p if the associated flow rule applies. Using the above, Prager obtained the plastic strain increment as

$$d\epsilon_{ij}^{(p)} = \frac{9H(f))}{4Ck^2} < (\sigma'_{kl} - \alpha'_{kl})d\sigma_{kl} > (\sigma'_{ij} - \alpha'_{ij})$$

where $C \approx \frac{\sigma_u - \sigma_Y}{\epsilon_u}$.

To eliminate some inconsistencies found with the linear rule, Ziegler proposed the modified evolution equation

$$\dot{\alpha}_{ij} = \dot{\mu}(\sigma_{ij} - \alpha_{ij})$$

where $\dot{\mu} > 0$.

Nonlinear evolution equations of the form

$$\dot{\alpha}_{ij} = ce_{ij}^{(p)} - \gamma\alpha_{ij}\dot{e}_{eff}^{(p)}$$

have also been proposed.

7.9 Combined Isotropic and Kinematic Hardening

The flow rule in this case can be expressed as

$$f(\sigma_{ij} - \alpha_{ij}, \kappa) = 0$$

where α_{ij} and κ are the internal variables. If a nonlinear evolution equation is assumed for α_{ij}, κ is assumed a simple monotonically increasing function of the effective plastic strain and the material is assumed to follow an associated flow rule, it can be shown that

$$K^p = c - \sqrt{\frac{2}{3}}\gamma\alpha_{ij}n_{ij} - \sqrt{\frac{2}{3}}\frac{\partial f}{\partial \kappa}\frac{\partial \kappa}{\partial e^{(p)}_{eff}}(\frac{\partial f}{\partial \sigma_{ij}}\frac{\partial f}{\partial \sigma_{ij}})^{-1/2}$$

and

$$\dot{e}^{(p)}_{ij} = \frac{n_{ij}n_{kl}\dot{\sigma}_{kl}}{K^p}$$

and

$$de_{rq} = [C_{ijkl} + \frac{n_{kl}n_{rq}}{K^p}]d\sigma_{kl}$$

thus allowing determination of the plastic strain rate and the stress-strain relationship.

7.10 Hardening due to Arbitrary Loadings

The above isotropic, kinematic and combined hardening models are adequate for the description of hardening under monotonic loadings. Hardening behavior becomes more complex under cyclic or under arbitrary loading conditions.

Specifically, for the case of cyclic loading the following aspects have been identified:

- Plastic strain accumulates in the form of a ratcheting effect in the direction of the mean stress;
- Mean stress gradually relaxes to zero for strain cycles with nonzero mean;
- Smooth elasto-plastic transition;
- Material tends to a stabilized state under kinematic hardening;
- Extensive plastic loading erases most of the previous effects.

Mroz made use of multiple yield surfaces with combined isotropic and kinematic hardening to account for the effect of cyclic loading. In Mroz model, the plastic modulus varies in a piecewise constant manner between any two surfaces. Dafalias and Popov proposed a model for cyclic hardening involving a yield surface and a bounding surface with the plastic modulus allowed to vary continuously between the two surfaces. Finally, Valanis proposed a theory of plasticity without a yield surface.

8 Finite Elements in Plasticity

The application of finite element methods to the solution of inelastic deformation problems is an extension of the elastic deformation algorithm. Effectively, one solves a selected sequence of elastic problems that converges to the solution of the desired inelastic problem.

The fundamental equation of the finite element method, constituting the expression for the condition of mechanical equilibrium can be expressed in the following generalized form

$$\mathbf{Q} = \mathbf{F}$$

where $\mathbf{Q}$ is the global internal force matrix and $\mathbf{F}$ is the global matrix of discretized external loads. These matrices are obtained by summing together the contributions due to the local or element matrices.

For the linear elastic body, the specific form assumed by the above is

$$\mathbf{K}\mathbf{u} = \mathbf{F}$$

where $\mathbf{K}$ is the global stiffness matrix given by

$$\mathbf{K} = \sum_e \mathbf{A}^{eT}\mathbf{K}^e\mathbf{A}^e$$

with $\mathbf{A}^e$ is the Boolean matrix such that $A^e_{mn} = 1$ if the m-th elemental degree of freedom corresponds to the n-th global degree of freedom and zero otherwise. $\mathbf{K}^e$ is the element stiffness matrix defined by

$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^{eT}\mathbf{C}\mathbf{B}^e dV$$

where $\mathbf{B}^e$ is the matrix containing the derivatives of the shape functions and C is the elasticity matrix containing the appropriate material properties. Finally, $\mathbf{u}$ is the vector of nodal displacements.

In the case of rate independent inelastic deformation, the constitutive equation of the deforming medium has the generic rate form

$$\dot{\sigma} = \mathbf{C}_{inel}\dot{\epsilon}$$

where the inelastic modulus matrix $\mathbf{C}_{inel}$ is a nonlinear operator since it depends on ϵ as well as on σ, ξ and the temperature T. Specifically,

$$\mathbf{C}_{inel} = \mathbf{C}$$

if $f < 0$ or $\dot{\lambda} \leq 0$ and

$$\mathbf{C}_{inel} = \mathbf{C}_{el} - \frac{1}{L}\mathbf{C}_{el}\mathbf{g}\frac{\partial f}{\partial \sigma_{ij}}\mathbf{C}_{el}$$

if $f = 0$ and $\dot{\lambda} > 0$

This must be suplemented by the internal variable evolution equation, a suitable form of which is

$$\dot{\xi} = \dot{\lambda}\mathbf{h}(\sigma, \xi)$$

Typically, $\mathbf{h}$ is a known function (e.g. the unit normal to the yield function in the case of the associated flow rule) and the internal variable matrix ξ includes ϵ^p and hence the above evolution equation for it includes the flow rule equation

$$\dot{\epsilon}^p = \dot{\lambda}\mathbf{g}(\sigma, xi)$$

Here

$$\dot{\lambda} = \frac{< f* >}{H}$$

where

$$f* = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}$$

and

$$H = -\sum_{\alpha} \frac{\partial f}{\partial \xi_{alpha}} \dot{\xi}_{alpha}$$

and f is the yield function.

However, the constitutive behavior of the deforming material can also be expressed as

$$\dot{\sigma}_{ij} = C_{ijkl}(\dot{\epsilon}_{kl} - \dot{e}psilon^p_{kl}) = \dot{\sigma} = C(\dot{\epsilon} - \dot{\epsilon}^p)$$

where ϵ_{kl} and ϵ^p_{kl} are, respectively, the total strain and the inelastic (i.e. permanent) strain.

The explicit Euler method can be used to integrate numerically and advance the calculation from time level t to $t + \Delta t$ yielding

$$\Delta\xi = \Delta\lambda\mathbf{h}$$

$$\Delta\epsilon^p = \Delta\lambda\mathbf{g}$$

where $\mathbf{h}$ (which includes $\mathbf{g}$ is evaluated at the start of the time step. The problem of conditional stability of the explicit method requires using sometimes exceedingly small time steps and implicit numerical integration methods are preferred.

The global stiffness matrix may be defined in terms of $\mathbf{C}_{inel}$ as for the elastic case, i.e.

$$\mathbf{K}_{inel} = \sum_e \mathbf{A}^{eT}[\int_{V^e} \mathbf{B}^{eT}\mathbf{C}\mathbf{B}^e dV]\mathbf{A}^e$$

Given an imposed external load history, one solves the elasto-plastic problem in a step-by step manner by computing at each step the incremental displacements resulting from an incremental applied load. The computation starts from a given initial state by first assuming that $\dot{\lambda}$ in the next step has the same sign as in the current one. Then one computes first the nodal velocities $\dot{\mathbf{u}}$ by solving

$$\mathbf{K}_{inel}\dot{\mathbf{u}} = \dot{\mathbf{F}}$$

and calculates the associated displacements, and the strain and stress fields. If one finds that the assumption about $\dot{\lambda}$ is correct, one can proceed with the next stage of the calculation.. If not, then one simply iterates modifying $\mathbf{K}_{inel}$ until the assumption is consistent with the result.

Once the nodal velocities are known, the displacement increments $\Delta\mathbf{u}$ resulting from imposing an increment of load $\Delta\mathbf{F}$ are computed by iteration (usually Newton-Raphson) as follows

$$\Delta\mathbf{u}^{(k+1)} = \Delta\mathbf{u}^{(k)} + \mathbf{K}_{inel}^{-1}\Delta\mathbf{R}^{(k)}$$

where $\Delta\mathbf{R}^{(k)}$ are residual forces resulting from the inequality between the incremental internal forces and the incremental external load.

Once some suitable norm of the residual forces becomes sufficiently small, the incremental displacements are accepted and the calculation can proceed in the same manner to the determination of the subsequent increments.

9 Examples

9.1 A Constitutive Equation Incorporating Combined Hardening

Bamman proposed a constitutive equation capable of incorporating temperature and strain rate effects. In Bamman's model, the plastic deformation contribution to the strain rate is assumed to result from a balance between temperature dependent hardening and recovery processes. For the special case of uniaxial stress σ , this is assumed to be of the form

$$\dot{\epsilon}^p = f(T)\sinh[\frac{|\sigma - \alpha| - \kappa - Y(T)}{V(T)}]$$

where T is the temperature, α is a tensorial hardening variable, κ is a scalar hardening variable, and

$$f(T) = C_5\exp(-\frac{C_6}{T})$$

$$Y(T) = C_3\exp(\frac{C_4}{T})(\frac{1}{2}[1 + \tan(C_{19}(C_{20} - T))])$$

$$V(T) = C_1 \exp(-\frac{C_2}{T})$$

Further

$$\dot{\alpha} = h\mu(T)\dot{\epsilon}^p - [r_s(T) + r_d(T)|\dot{\epsilon}^p|]\alpha^2 sign(\alpha)$$

$$\dot{\kappa} = H\mu(T)\dot{\epsilon}^p - [R_s(T) + R_d(T)|\dot{\epsilon}^p|]\kappa^2$$

with

$$\mu(T) = \mu(T_0)[1 - (\frac{T}{T_m})\exp(T_h(1 - \frac{T}{T_m})]$$

where T_m is the melting temperature and T_h is the characteristic homologous temperature.

The quantities h, r_s and r_d are hardening and recovery parameters associated with the tensor variable α and the quantities H, R_s and R_d are hardening and recovery parameters associated with the scalar variable κ. The hardening and recovery parameters are assumed to be related to the temperature according to

$$r_d(T) = C_7 \exp(-\frac{C_8}{T})$$

$$h(T) = C_9 - C_{10}T$$

$$r_s(T) = C_{11} \exp(-\frac{C_{12}}{T})$$

$$R_d(T) = C_{13} \exp(-\frac{C_{14}}{T})$$

$$H(T) = C_{15} - C_{16}T$$

$$R_s(T) = C_{17} \exp(-\frac{C_{18}}{T})$$

The stress is assumed implicitly given by

$$\dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}^p)$$

The integrated form of the uniaxial stress-strain equation is

$$\sigma = \beta + \sqrt{\frac{h\dot{\epsilon}}{r_d\dot{\epsilon} + r_s}} \tanh \sqrt{\frac{hr_d\dot{\epsilon} + r_s}{\dot{\epsilon}}}\epsilon$$

where

$$\beta(\dot{\epsilon}, T) = Y(T) + V(T)\sinh^{-1}[\frac{|\sigma - \alpha|}{f(T)}]$$

The purpose of modeling in this case is to derive values of the constants C_i , $i = 1, 2, ..., 20$ from given experimental data about ϵ, $\dot{\epsilon}$ and σ so as to be able to use the given constitutive relationship in metalworking deformation investigations as well as in lifing analyzes.

For the determination of the values of the constants C_i one can use a multi parameter nonlinear equation fitting method. This is the method of nonlinear least squares and good quality, robust routines are available. One such routine, MINPACK can be applied. The method proceeds by minimizing the goal function representing the sum of the squares of the differences between the experimental values and the values predicted from the constitutive equation, i.e. the method seeks to determine the values of the coefficients C_i that minimize the sum

$$S(\mathbf{C}) = \sum_{j=1}^{N}[f(\mathbf{C}, x_j) - y_j]^2$$

where N is the total number of data points (x_j, y_j) and $\mathbf{C}$ represents the vector of unknown parameters C_i. MINPACK implements a descent scheme first proposed by Levenberg and Marquardt. If $\mathbf{x}$ represents the vector of unknown parameters and an initial guess is bfx_c, an improved guess $\mathbf{x}_+$ is given by

$$\mathbf{x}_+ = \mathbf{x}_c - [J(\mathbf{x}_c)^T J(\mathbf{x}_c) + \mu_c \mathbf{I}]^{-1} J(\mathbf{x}_c)^T R(\mathbf{x}_c)$$

where $\mu_c = 0$ if $\delta_c \geq ||(J(\mathbf{x}_c)^T J(\mathbf{x}_c))^{-1} J(\mathbf{x}_c)^T R(\mathbf{x}_c)||_2$ and zero otherwise. Various versions of the method exist based on the criteria used to select the values of μ_c and δ_c.

9.2 Elastic-Plastic Deformation of a Spherical Shell

A simple illustration of the methods of structural elastic plastic equilibrium analysis is provided by the problem of the hollow spherical shell (inner radius a, outer radius b), under internal pressure p. For simplicity, the material of the sphere will be assumed elastic-perfectly plastic (i.e. no work hardening).

The equilibrium equation in this case becomes

$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\phi}{r} = 0$$

This must be solved subject to the conditions

$$\sigma_r = -p$$

at $r = a$, and

$$\sigma_r = 0$$

at $r = b$.

Further, the components of strain

$$\epsilon_r = \frac{du}{dr}$$

$$\epsilon_\phi = \frac{u}{r}$$

where u is the radial displacement, must satisfy the continuity condition

$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\phi}{r} = 0$$

The problem is readily solved if deformation is elastic, the condition prevailing at relatively small values of p. Assuming incompressible behavior the solution becomes

$$\sigma_r = p\frac{(a/b)^3 - (a/r)^3}{1 - (a/b)^3}$$

$$\sigma_\phi = p\frac{(a/b)^3 + \frac{1}{2}(a/r)^3}{1 - (a/b)^3}$$

and

$$u = \frac{p}{E}\frac{1.5b^3/2r^2}{(b/a)^3 - 1}$$

As the pressure increases, yielding will take place when the yielding criterion is satisfied. Specifically, if the Tresca criterion is used, the yielding condition is

$$\sigma_\phi - \sigma_r = \sigma_Y$$

In terms of pressure p_Y, the yield condition is

$$p_Y = \frac{2\sigma_Y}{3}(1 - \frac{a^3}{b^3})$$

Once yielding takes place, if the internal pressure increases further, a plastic region will grow in size from the initial inner radius a. Eventually, at sufficiently high pressures, the plastic zone will emerge at the outer radius and the whole shell will be in the plastic state.

However, for intermediate pressures, elastic and plastic regions will coexist with the boundary between them located at some radius $r = c$.

The equilibrium equation can be written as

$$\frac{d\sigma_r}{dr} - 2\frac{\sigma_Y}{r} = 0$$

and this must be solved taking into account that at the elastic-plastic radius $r = c$, $\sigma_r = -p_Y$.

Integrating the equilibrium equation and introducing the additional condition gives the following solution for the stresses within the plastic zone,

$$\sigma_r = \frac{2\sigma_Y}{3}[3\ln(\frac{r}{c}) - 1 + (\frac{c}{b})^3]$$

and

$$\sigma_\phi = \frac{2\sigma_Y}{3}[3\ln(\frac{r}{c}) + \frac{1}{2} + (\frac{c}{b})^3]$$

while the stresses within the elastic region become

$$\sigma_r = \frac{2\sigma_Y}{3}[(\frac{c}{b})^3 - (\frac{c}{r})^3]$$

and

$$\sigma_\phi = \frac{2\sigma_Y}{3}[(\frac{c}{b})^3 + \frac{1}{2}(\frac{c}{r})^3]$$

The radius of the elastic-plastic boundary must be obtained by solving the following transcendental equation

$$p = \frac{2\sigma_Y}{3}[3\ln(\frac{c}{a}) + 1 - (\frac{c}{b})^3]$$

Once the pressure reaches the value

$$p = \sigma_Y \ln(\frac{b}{a})^2$$

plasticity has extended through the entire thickness of the spherical shell and the stresses become

$$\sigma_r = \sigma_Y \ln(\frac{r}{b})^2$$

and

$$\sigma_\phi = \sigma_Y[1 + \ln(\frac{r}{b})^2]$$

9.3 Elastic-Plastic Deformation of a Pressurized Thick-Walled Tube

A thick-walled tube (inner radius a, outer radius b) is pressurized internally by pressure p while maintained at zero pressure outside. As the internal pressure increases, the material of the tube yields first at $r = a$ and under even larger pressure, the elastic-plastic boundary at $r = c$ moves from the inner to the outer radius. At any intermediate pressure, the tube wall consists of an inner plastic zone surrounded by an elastic zone. The relationship between the internal pressure and the position of the elastic-plastic boundary (assuming no strain hardening and that Tresca's yield criterion $\sigma_\theta - \sigma_r = \sigma_Y$ applies) is given by

$$\frac{p}{\sigma_Y} = \ln(\frac{c}{a}) + \frac{1}{2}[1 - (\frac{c}{b})^2)$$

In the outer zone, the stresses are elastic and are given by

$$\sigma_r = \sigma_Y \frac{c^2}{2b^2}[1 - (\frac{b}{r})^2]$$

and

$$\sigma_\theta = \sigma_Y \frac{c^2}{2b^2}[1 + (\frac{b}{r})^2]$$

In the inner zone, the material is in the plastic state and the stresses are

$$\sigma_r = \sigma_Y[-\frac{1}{2} - \ln(\frac{c}{r}) + \frac{c^2}{2b^2}]$$

and

$$\sigma_\theta = \sigma_Y[\frac{1}{2} - \ln(\frac{c}{r}) + \frac{c^2}{2b^2})]$$

9.4 Drawing of a Wide Strip through Wedge Shaped Dies

Concepts of continuum plasticity are useful in the analysis of metal forming operations. One such example is now considered. Metals strips can be plastically thinned by drawing them through wedge shaped dies. When drawing wide strips, plane strain conditions are obtained. A simple approach to the analysis of the process is provided by the slab method. In this method, the deformation is assumed homogeneous through the thickness of the strip and the governing equation is obtained by performing differential force balances.

Let the original strip be of thickness h_b and let the thickness after drawing be h_a. Let its width be w and the angle of the dies with the drawing direction α. Finally, let p be the pressure normal to the die surface and μ the coefficient of friction at the die-strip interface. Consider an element of strip of length dx along the drawing direction. A differential force balance along the drawing direction involves the longitudinal stress, the die pressure and the frictional force (on both dies), i.e.

$$\sigma_x dh + h d\sigma_x + 2pdx \tan\alpha + 2\mu p dx = 0$$

which readily reduces to

$$h d\sigma_x + [\sigma_x + p(1+B)]dh = 0$$

where $B = \mu \cot\alpha$ and $h = 2x \tan\alpha$ inside the dies.

A differential force balance along the normal direction gives

$$\sigma_y dx + pdx - \mu p \tan\alpha dx = 0$$

Since μ is usually small, the last term is negligible and one can assume that $\sigma_y = -p$

The von Mises yield criterion under plane strain conditions is

$$\sigma_1 - \sigma_3 = 2k = 1.155\sigma_Y$$

where the principal stresses are $\sigma_1 = \sigma_x$ and $\sigma_3 = -p$.

Introducing the value of p derived from the yield condition into the x force balance and rearranging gives

$$\frac{d\sigma_x}{dh} = \frac{B\sigma_x - 1.155\sigma_Y(1+B)}{h}$$

Assuming that α, μ and σ_Y are all constants and using the boundary condition $\sigma_x = 0$ at $h = h_b$ leads to

$$\sigma_x = 1.155\sigma_Y \frac{1+B}{B}[1 - (\frac{h}{h_b})^B]$$

and the drawing stress exerted by the drawbench grips (σ_x at $h = h_a$) is

$$\sigma_x|_{h=h_a} = 1.155\sigma_Y \frac{1+B}{B}[1 - (\frac{h_a}{h_b})^B]$$

10 Exercises

10.1

Use the Ramberg-Osgood equation to produce stress-strain plots for the materials in the following table, (Room temperature data; all in consistent units).

Material	$\sigma_Y(MPa)$	M_Y	K_Y
$316SS$	133	4.5	435
$IN-100$	650	5.6	655
$Ti-64$	300	4.3	884
$IN-718$	500	5.2	1658

10.2

The density of dislocations in a well annealed sample of 316 Stainless Steel is of the order of $10^{12} m^{-2}$. As the sample is strained under uniaxial tension, the dislocation density increases and so does the yield stress (of the deformed material). Estimate the increase in dislocation density with strain. Assume $|\mathbf{b}| \approx 10^{-9} m$.

10.3

The following state of stress exists at a point in a well annealed titanium alloy component: $\sigma_x = 600$ MPa, $\sigma_y = 200$ MPa, $\sigma_z = 100$ MPa, and $\tau_{xy} = 60$ MPa. If the uniaxial yield stress of the alloy is $\sigma_Y = 800$ MPa, would the component yield under the imposed stress state?

10.4

Consider a long, thick walled (inner radius a, outer radius $b = 2a$) pressurized tube.

a.- At relatively low values of the internal pressure p the behavior is elastic and the radial and azimuthal stresses are given by

$$\sigma_r = p \frac{a^2}{b^2 - a^2} [1 - (\frac{b}{r})^2]$$

and

$$\sigma_\theta = p \frac{a^2}{b^2 - a^2} [1 + (\frac{b}{r})^2]$$

Rearrange the equations into dimensionless form and then plot the ratios σ_r/p and σ_θ/p versus the ratio r/b.

b.- At higher pressures, the material of the tube yields first at $r = a$ and with increasing pressure the elastic-plastic boundary at $r = c$ moves from the inner to the outer radius. At any intermediate pressure, the tube wall consists of an inner plastic zone surrounded by an elastic zone. The relationship between the internal pressure and the position of the elastic-plastic boundary (assuming no strain hardening and that Tresca's yield criterion $\sigma_\theta - \sigma_r = \sigma_Y$ applies) is given by

$$\frac{p}{\sigma_Y} = \ln(\frac{c}{a}) + \frac{1}{2}[1 - (\frac{c}{b})^2)$$

In the outer zone, the stresses are elastic and are given by

$$\sigma_r = \sigma_Y \frac{c^2}{2b^2} [1 - (\frac{b}{r})^2]$$

and

$$\sigma_\theta = \sigma_Y \frac{c^2}{2b^2}[1 + (\frac{b}{r})^2]$$

In the inner zone, the material is in the plastic state and the stresses are

$$\sigma_r = \sigma_Y[-\frac{1}{2} - \ln(\frac{c}{r}) + \frac{c^2}{2b^2}]$$

and

$$\sigma_\theta = \sigma_Y[\frac{1}{2} - \ln(\frac{c}{r}) + \frac{c^2}{2b^2})]$$

Compute the value of p/σ_Y for which $c = \frac{3}{4}b$, then rearrange the stress equations into dimensionless form and plot the ratios σ_r/σ_Y and σ_θ/σ_Y versus the ratio r/b.

10.5

A simply supported beam has length $L = 1$, breadth $b = 1$ and height $h = 0.1$. The elastic modulus is $E = 10^{11}$ and Poisson's ratio is $\nu = 0.3$. The stress-strain relationship for the pastic deformation of the beam, obtained from a simple tension test, is summarized in the following table:

$e(-)$	$\sigma(Pa)$
0.001	100×10^6
0.02	300×10^6
0.04	500×10^6
0.06	700×10^6
0.08	750×10^6
0.10	775×10^6
0.20	780×10^6

In the middle of the upper edge of the beam, a downward point force F is applied. If the applied force is relatively small, the beam will deform elastically. As the force increases, incipient plastic deformation will start at the most highly stressed points in the beam. Eventually, a plastic hinge will appear in the middle of the beam. Use the Ansys finite element software with linear elements to determine the force required for incipient plastic deformation and for the formation of the plastic hinge. Provide evidence of the validity of your results.

10.6

A cantilever beam has length $L = 1$, breadth $b = 1$ and height $h = 0.1$. The elastic modulus is $E = 10^{11}$ and Poisson's ratio is $\nu = 0.3$. The stress-strain relationship for the pastic deformation of the beam, obtained from a simple tension test, is summarized in the following table:

$e(-)$	$\sigma(Pa)$
0.001	100×10^6
0.02	300×10^6
0.04	500×10^6
0.06	700×10^6
0.08	750×10^6
0.10	775×10^6
0.20	780×10^6

At the upper edge of the free end, an upward load F is applied. If the applied force is relatively small, the beam will deform elastically. As the force increases, incipient plastic deformation will start at the most highly stressed points in the beam. Eventually, a plastic hinge will appear in the middle of the beam. Use the Ansys finite element software with linear elements to determine the force required for incipient plastic deformation and for the formation of the plastic hinge. Provide evidence of the validity of your results.