

Chapter 6

Creep/Viscoplasticity

1 Introduction

Creep is the inelastic response of materials loaded at high temperatures. Temperature has important effects on deformation phenomena since microstructural defect rearrangement processes are often accelerated at high temperatures. Since these processes tend to soften the material they counteract the strain hardening produced by plastic deformation. Creep behavior resembles viscoelasticity since the resulting strain at constant applied stress is a function of time. However, in contrast with viscoelastic behavior, permanent deformation remains following creep. As a result, creep behavior is also frequently known as viscoplasticity. These two terms will be used here interchangeably.

Elasticity and plasticity are mechanical responses to loading which are independent of time. As soon as the load is applied, the corresponding level of strain sets in. In contrast, during creep the mechanical response is time dependent. This is somewhat analogous to viscoelastic behavior except that during creep an often significant portion of the strain is permanent and remains after unloading.

With appropriate modifications, the standard tension test can be applied to investigate the mechanical response of materials deforming at high temperatures. Easiest to perform are tension tests at constant load and temperature. The resulting strain is measured as a function of time. This is often called the creep test and the resulting strain-time curves are called creep curves. If the creep test extends over a sufficiently long period of time, a stage may be reached at which the sample weakens rapidly and eventually breaks. Tests aimed at investigating this final stage of the creep process are called creep rupture tests.

By appropriate modifications of the creep testing equipment, other useful tests can be developed such as the constant stress tests and the constant strain rate test.

Using the results of creep tests, another manifestation of creep response has been observed; namely, the relaxation of stress over time. If a creeping material is strained to a specified level and the strain is then maintained fixed at that level, the stress obtained decays from its initial value.

Cyclic loading is often associated with material fatigue processes which eventually may cause the failure of the loaded component. At high temperatures, creep phenomena appear

and they have an important influence on the fatigue process. Under cyclic loading, the resulting creep deformation and relaxation phases affect the fatigue response of the material depending on their relative duration according to the specific form of the cycling pattern.

Typical strain-time curves obtained from creep tests at constant load over sufficiently long periods of time often exhibit three characteristic stages. These are:

- Primary or Transient Creep. Following the setting in of the instantaneous elastic strain ϵ_0 , the material deforms rapidly but at a decreasing rate. The duration of this stage is typically relatively short in relation to the total creep curve.
- Secondary or Steady-State Creep. Here, the creep strain rate reaches a minimum value and remains approximately constant over a relatively long period of time.
- Tertiary Creep. In this stage, the creep strain rate accelerates rapidly.
- Rupture. The material is unable to withstand the load anymore and breaks.

2 Mathematical Modeling of Creep

While the creep response of materials is intimately related to microstructural processes that take place inside the material during deformation, a continuum description of the creep process has proven to be of great engineering usefulness. From the standpoint of the continuum representation, solution of the creep problem requires statement of the mechanical equilibrium equations, appropriate constitutive equations for creep behavior and suitable initial and boundary conditions. Since temperature is a key variable in creep the energy balance equation may have to be solved also.

The simplest case to investigate is uniaxial loading. A generic formal mathematical representation of the creep curve is

$$\epsilon^c = F(\sigma, T, t) = f(\sigma)g(T)h(t)$$

where ϵ^c is the creep strain, σ is stress, T is temperature and t is time. Note that time and temperature are explicitly included in the representation since these two variables play key roles in the creep response. The introduction of the functions f , g and h above implies the assumption of separability of the effects of stress, temperature and time, which is frequently made. The above expression is the basis of the equation of state formulation used to investigate creep phenomena under variable stress. In this formulation the response depends on present state explicitly and it is in contrast with memory formulations that are somewhat less well developed.

3 Creep Rate Laws and the Hardening Formulations

In the design for structural integrity one is often most interested in the primary and secondary stages of creep. A commonly used equation of state representation of these two creep stages is provided by the Bailey-Norton law, sometimes called power law creep law. This is given by

$$\epsilon^c = A\sigma^n t^m$$

The above is often expressed in rate form as

$$\dot{\epsilon}^c = A\sigma^n m t^{m-1}$$

and is called the time hardening formulation of power law creep, where A , m , n are temperature dependent material constants.

If the Bailey-Norton law is solved for t and the result substituted into the above, the result is

$$\dot{\epsilon}^c = A^{1/m} m \sigma^{n/m} (\epsilon^c)^{(m-1)/m}$$

This is called the strain hardening formulation of power law creep.

The time and strain hardening formulations are used in practice to predict the creep curve under a variable stress history using only data obtained from multiple creep tests, each at constant stress. Experience indicates that the strain hardening formulation often produces better agreement with the results of actual tests under variable stress.

Several other creep rate laws have been proposed, they include Andrade's law

$$\dot{\epsilon}^c = \frac{\beta}{3} \frac{1}{t^{2/3} + \beta t} + k$$

where β and k are constants; the exponential law

$$\dot{\epsilon}^c = \dot{\epsilon}_e \exp\left(\frac{\sigma}{\sigma_e}\right)$$

where $\dot{\epsilon}_e$ and σ_e are constants; the hyperbolic sine law

$$\dot{\epsilon}^c = 2\dot{\epsilon}_e \sinh\left(\frac{\sigma}{\sigma_e}\right)$$

A generic representation of all the above creep laws is

$$\dot{\epsilon} = v(\sigma)$$

where $v(\cdot)$ is the specific functional form used.

Another set of creep laws have been proposed not in terms of the creep strain rate but the creep strain as follows

$$\epsilon^c = A(\sigma)[1 - \exp(-r(\sigma)t)] + \dot{\epsilon}_m t$$

$$\epsilon^c = \frac{a_1 t}{1 + b_1 t} + \dot{\epsilon}_m t$$

$$\epsilon^c = A(\sigma)[1 - \exp(-r(\sigma)t)] + B(\sigma)[1 - \exp(-s(\sigma)t)] + \dot{\epsilon}_m t$$

$$\epsilon^c = \frac{a_1 t}{1 + b_1 t} + \frac{a_2 t}{1 + b_2 t} + \dot{\epsilon}_m t$$

For creep problems at constant stress the above expressions can be used as given. However, in practice one is often interested in solving creep problems in structural members subjected to time varying stresses. In those case, the procedures suggested by either the time hardening or strain hardening formulations must be used to determine the actual creep response from the individual laws given above. A commonly used method to deal with stress reversals consists in redefining the strain hardening measure relative to reference creep strains.

4 Microstructure-based Creep Law for Steady State Creep

For steady state creep the strain rate has been shown to be well represented by the Mukherjee-Bird-Dorn equation, which expresses the creep rate in terms of stress, temperature and grain size as

$$\dot{\epsilon}_s^c = \frac{AGb}{kT} D \left(\frac{b}{d}\right)^p \left(\frac{\sigma}{G}\right)^n = A' \exp(Q/kT)$$

where A is a dimensionless constant, D is the diffusion coefficient given by

$$D = D_0 \exp(-Q/RT)$$

with D_0 is a frequency factor, Q is an activation energy, $R = N_A k$ is the gas constant N_A is Avogadro's number, k is Boltzmann's constant, d is grain size and p and n are material constants. To represent experimental data one plots $\dot{\epsilon} kT / D G b$ versus σ / G

5 Correlation and Extrapolation Methods

The creep life of structural components can be determined from the standard creep curve. However, this may require impractically long testing times and there has long been a need to be able to use results of accelerated testing at extreme conditions to predict creep life at less extreme conditions.

Extrapolation methods work by establishing correlations involving temperature, time to rupture and stress such that the least possible number of accelerated tests are used to determine life under less extreme conditions. Various approaches are commonly used:

- Larson-Miller
- Manson-Haferd
- Sherby-Dorn
- Monkman-Grant

In the Larson-Miller procedure, the results of constant engineering stress tests are used and a parameter m_{LM} is defined by

$$m_{LM} = T(\log t_r + C)$$

where T is temperature (K), t_r is time to rupture (hrs) and C is a material constant. So, if the material constant C is known, to determine the rupture time under constant stress at some temperature, an accelerated test at the same stress but higher temperature is carried out to determine m_{LM} . The desired rupture time at the higher temperature can then be readily determined

The corresponding parameters for the other approaches mentioned include:

- The Manson-Haferd parameter m_{MH} ,

$$m_{MH} = \frac{\log t_r - \log t_a}{T - T_a}$$

- The Orr-Sherby-Dorn parameter m_{OSD}

$$m_{OSD} = \ln t_r - \frac{Q}{kT}$$

- The Monkman-Grant relationship between steady state creep rate and rupture time

$$\dot{\epsilon}_s^c = \frac{k'}{t_r}$$

but since $\dot{\epsilon}_s = A' \exp(Q/kT)$

$$t_r = \frac{k'}{A'} \exp(Q/kT)$$

or, equivalently

$$\log t_r - m_{MG} = 0.43 \frac{Q}{kT}$$

where $m_{MG} = k'/A'$

6 Creep Mechanisms

Atoms in solids vibrate at high frequencies ($\approx 10^{13} s^{-1}$) about their mean positions. Occasionally ($\approx 1/10^6$ times), the amplitude of the vibration is large enough for the atom to move out of its current location and into a neighboring site. This process is called solid state diffusion. Diffusion is a thermally activated process. Thermal energy provides the necessary activation energy required to overcome the potential energy barrier preventing atomic displacements. Therefore, solid state diffusion processes are enhanced significantly at high temperatures. Since atomic movements are directly related to microstructural reorganization processes it is then natural to expect that creep phenomena will be directly related to solid state diffusion phenomena, particularly, at the highest temperatures.

If the temperature is relatively low, diffusion is somewhat less important and the creep deformation process at the atomic scale is most influenced by slip and glide phenomena characteristic of low temperature plastic deformation.

Therefore, according to the prevailing temperature and stress in a creeping solid, various different microstructural mechanisms determine the observed creep behavior. A summary of the situation is as follows

6.1 Solid State Diffusion Dominated Creep

At the highest temperatures, solid state diffusion is dominant and one has diffusional creep. Depending on the stress level, from lowest to highest one has the following regimes,

- Nabarro-Herring or Bulk Diffusion Creep. At relative low stresses, the rate of solid state diffusion in the bulk of crystal grains determines the creep rate. The following relationship for the creep rate was first obtained by Nabarro and Herring,

$$\dot{\epsilon} = A_{NH} \frac{D G b}{kT} \left(\frac{b}{d}\right)^2 \left(\frac{\sigma}{G}\right)$$

where $A_{NH} \approx 12.5$, D is the diffusion coefficient in the bulk of the grains and d is the grain size.

- Coble or Grain Boundary Diffusion Creep. As the stress increases and/or the temperature decreases, the rate of solid state diffusion along grain boundaries becomes more intense than in the bulk of the grains and hence it becomes determinant of the creep strain. The following relationship for the creep rate was first obtained by Coble,

$$\dot{\epsilon} = A_C \frac{D_{gb} G b}{kT} \left(\frac{\delta}{b}\right) \left(\frac{b}{d}\right)^3 \left(\frac{\sigma}{G}\right)$$

where $A_C \approx 40$, D_{gb} is the diffusion coefficient in the grain boundary of the grains and δ is the grain boundary thickness.

- Harper-Dorn or Dislocation Climb Creep. At still lower stresses but higher temperatures and for large grain sizes, extensive diffusion assisted dislocation climb has sometimes been found to be determinant of the creep rate. The following relationship for the creep rate was first obtained by Harper and Dorn,

$$\dot{\epsilon} = A_{HD} \frac{D G b}{kT} \left(\frac{\sigma}{G}\right)$$

where $A_{HD} \approx 10^{-11}$.

6.2 Glide or Sliding Dominated Creep

At relatively lower temperatures or higher stresses, glide processes begin to dominate the creep response. However, the diffusional contribution may still be significant. The following regimes have been observed,

- Dislocation Creep or Vacancy Diffusion Assisted Glide. At relatively high stresses and temperatures dislocation glide over slip planes may become assisted by diffusion. Specifically, obstacles to dislocation motion can be overcome with the assistance of vacancy diffusion around the obstacle. The following relationship for the creep rate was first obtained by Weertman,

$$\dot{\epsilon} \approx \frac{D G b}{kT} \left(\frac{\sigma}{G}\right)^5$$

Note the similarity with the Mukherjee-Bird-Dorn formula.

- Dislocation Glide Creep. As the stress rises to the highest levels then the processes of dislocation slip over selected crystallographic planes and directions become most important and the influence of diffusion is relatively small. The deformation mechanisms is essentially the same prevailing at room temperature.
- Grain Boundary Sliding Creep. At the highest stresses and temperatures, individual grains begin to slide over each other. The structural integrity of the component is then seriously compromised as tertiary creep sets in.

The various observed creep mechanisms have been summarized by a number of materials of engineering interested into graphical constructs called deformation mechanism maps. A map is a two dimensional representation where values of the homologous temperature T/T_M , where T_M is the melting point of the material are plotted along the horizontal axis and values of the normalized stress σ/G , where G is the shear modulus are plotted along the vertical axis. Regions of predominance of the individual creep mechanisms are indicated in the graph in terms of the ranges of values of stress and temperature for which each individual mechanism dominates the creep process. Additionally, iso-creep strain rate lines are superimposed for convenience. The maps are useful since they allow at a glance to obtain a semi-qualitative picture of the prevailing creep conditions under given stress and temperature.

7 Creep Equations for Multi axial Loading

In most engineering applications one is interested in investigating the creep behavior of components under multiaxial loading conditions. Therefore in order to be useful in practice the previously presented constitutive equations must be generalized to those conditions. Key requirements of a multiaxial loading formulation include:

- The formulation must reduce to uniaxial formulation under uniaxial loading.
- Incompressibility.
- Creep strain independent of hydrostatic stress.
- Coincidence of principal directions of stress and strain in isotropic materials.

If the creep strain rate tensor components are assumed proportional to the deviator tensor components the above requirements are all satisfied. Therefore

$$\dot{\epsilon}_{ij}^c = \lambda \sigma'_{ij}$$

where $\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$. The factor of proportionality λ is determined with the help of the effective stress and the effective creep strain rate as follows.

The effective stress is defined as

$$\begin{aligned} \sigma_e &= \frac{1}{\sqrt{2}} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{33} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2)]^{1/2} = \\ &= \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \sqrt{3J_2} \end{aligned}$$

where J_2 is the second invariant of the stress deviator tensor.

The effective creep strain rate is defined as

$$\begin{aligned}\dot{e}_e^c &= \frac{\sqrt{2}}{3} [(\dot{e}_{11}^c - \dot{e}_{22}^c)^2 + (\dot{e}_{33}^c - \dot{e}_{22}^c)^2 + (\dot{e}_{11}^c - \dot{e}_{33}^c)^2 + 6((\dot{e}_{12}^c)^2 + (\dot{e}_{23}^c)^2 + (\dot{e}_{13}^c)^2)]^{1/2} = \\ &= \sqrt{\frac{1}{2} \dot{e}_{ij}^c \dot{e}_{ij}^c} = \sqrt{\frac{4}{3} I_2}\end{aligned}$$

where I_2 is the second invariant of the creep strain rate tensor.

The parameter λ is thus given by

$$\lambda = \frac{3}{2} \frac{\dot{e}_e^c}{\sigma_e}$$

and can be determined from a uniaxial test.

Therefore, the Bailey-Norton law for the multiaxial case is now expressed in terms of effective stress and creep strain as

$$e_e^c = A \sigma_e^n t^m$$

so that

$$\lambda = \frac{3}{2} A m \sigma_e^{n-1} t^{m-1}$$

According to the time hardening rule, the creep rate is calculated as

$$\dot{e}_{ij}^c = \frac{3}{2} \sigma'_{ij} A m \sigma_e^{n-1} t^{m-1}$$

While for the strain hardening rule one uses

$$\dot{e}_{ij}^c = \frac{3}{2} \sigma'_{ij} m A^{1/m} \sigma_e^{(n/m)-1} (e_e^c)^{(m-1)/m}$$

For actual computation, it is convenient to introduce new parameters

$$\mu = \frac{1-m}{m}$$

$$n = \frac{n}{m}$$

and

$$\frac{1}{\tau \sigma_m^n} = m A^{1/m}$$

With these, the strain hardening rule above becomes

$$\dot{e}_{ij}^c = \frac{1}{\tau} \left(\frac{\sigma_e}{\sigma_m} \right)^n (e_e^c)^{-\mu} \frac{3}{2} \frac{\sigma'_{ij}}{\sigma_e} = \frac{3}{2} F \frac{\sigma'_{ij}}{\sigma_e}$$

where

$$F = \frac{1}{\tau} \left(\frac{\sigma_e}{\sigma_m} \right)^n (e_e^c)^{-\mu}$$

Thus

$$\dot{e}_e^c = \sqrt{\frac{1}{2} \dot{e}_{ij}^c \dot{e}_{ij}^c} = F = \frac{1}{\tau} \left(\frac{\sigma_e}{\sigma_m} \right)^n (e_e^c)^{-\mu}$$

which is equivalent to

$$\frac{d(e_e^c)^{1+\mu}}{dt} = \frac{1+\mu}{\tau} \left(\frac{\sigma_e}{\sigma_m} \right)^n$$

For the uniaxial loading case, integration of the above yields

$$e_e^c = \left[\frac{1+\mu}{\tau} \left(\frac{\sigma}{\sigma_m} \right)^n t \right]^{1/(1+\mu)}$$

which can then be used to fit the data of a creep test and determine the values of the parameters n , μ and $\tau \sigma_m^n$.

8 Formulation of the Creep Problem

The strain tensor components are assumed decomposable into thermo-elastic, plastic and creep contributions, i.e.

$$e_{ij} = e_{ij}^e + \alpha T \delta_{ij} + e_{ij}^p + e_{ij}^c$$

This is a set of six equations.

The individual contributions are given as follows. The elastic strain, by Hooke's law,

$$e_{ij}^e = \frac{\sigma'_{ij}}{2G}$$

another set of six equations.

The plastic strain, once the body has yielded, by the flow rule in terms of the plastic strain increments

$$de_{ij}^p = \frac{3}{2} \frac{de_e^p}{\sigma_e} \sigma'_{ij}$$

where $de_e^p = \sqrt{\frac{2}{3}de_{ij}^p de_{ij}^p}$, together with the experimental instantaneous stress-strain curve $e_e^p = e_e^p(\sigma_e)$. The above constitute a set of seven equations.

Finally, the creep strain is given by the creep flow rule

$$\dot{e}_{ij}^c = \frac{3}{2} \frac{\dot{e}_e^c}{\sigma_e} \sigma'_{ij}$$

together with the experimental creep curve $e_e^c = e_e^c(\sigma_e, T, t)$; another set of seven equations.

The above equations must be combined with the mechanical equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0$$

the strain-displacement relationships

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

and the energy equation

$$\dot{H} = \nabla \cdot (k \nabla T)$$

The three equilibrium equations together with the six strain-displacement relations and the energy equation produce ten additional equations.

The definitions of effective stress, effective plastic strain increment and effective creep strain increment yield three more equations and the stress and strain deviator component equations constitute twelve additional relationships.

There are thus a total of fifty one equations that can be used to determine the fifty one unknowns e_{ij} , e_{ij}^e , e_{ij}^p , e_{ij}^c , e'_{ij} , σ_{ij} , σ'_{ij} (six of each), u_i (three), and T , de_e^p , de_e^c , e_e^p , e_e^c , σ_e (one of each).

In practice, the solution of actual problems requires also specification of initial and boundary conditions.

9 Finite Elements in Creep/Viscoplasticity

The implementation of the finite element method for the solution of creep/ viscoplasticity problems is somewhat similar to that used in solving elasto-plastic problems. The additive decomposition of the strain tensor yields

$$\epsilon = \epsilon^e + \epsilon^i = \epsilon^e + \epsilon^c$$

where the plastic strain contribution is neglected. The constitutive equation is

$$\sigma = \mathbf{C}_{el}(\epsilon - \epsilon^c)$$

The finite element equation then becomes in this case

$$\mathbf{K}_c \mathbf{u} = \mathbf{F} - \mathbf{B}^T \mathbf{C} \epsilon^c$$

In the method of initial strains the computation proceeds in a step-by-step fashion.

- Set $t = 0$ and determine the elastic solution of the problem by setting $\epsilon^c = 0$. This defines the starting displacement, strain and stress fields.
- Assume the stress remains constant over a time step of size Δt and compute the increments in creep strain by

$$\Delta \epsilon_{ij}^c = \frac{3}{2} \frac{d\epsilon_{eff}^c}{dt} \frac{\sigma'_{ij}}{\sigma_{eff}} \Delta t$$

where

$$\sigma_{eff} = \sqrt{\frac{3}{2} \sigma'_{ij} \sigma'_{ij}}$$

and

$$epsilon_{eff} = \sqrt{\frac{2}{3} d\epsilon_{ij}^c d\epsilon_{ij}^c}$$

and using the experimentally determined creep curve in the form

$$\epsilon_{eff}^c = \epsilon_{eff}^c(\sigma_{eff}, T, t)$$

- Compute the new creep strain field at the end of the time step by

$$\epsilon_{ij}^c(t + \Delta t) = \epsilon_{ij}^c(t) + \Delta epsilon_{ij}^c$$

and then the stress field at the end of the time step from the constitutive relationship.

- If the stresses are less or equal to a preset fraction of the stresses at the previous time step, repeat the process to advance the calculation an additional time step, if not, decrease Δt and repeat until the stresses are less or equal to the preset fraction of the stresses at the previous time step.

10 Examples

10.1 Creep of a Pressurized Thick Walled Tube

A thick walled tube (inner radius a , outer radius b) is internally pressurized by pressure p while the pressure is zero outside. The equilibrium equation is

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

If the pressure is relatively low, the behavior is elastic and the stresses are given by

$$\sigma_r = p \frac{1 - (b/r)^2}{(b/a)^2 - 1}$$

$$\sigma_\theta = p \frac{1 + (b/r)^2}{(b/a)^2 - 1}$$

$$\sigma_z = p \frac{1}{(b/a)^2 - 1}$$

The incompressibility condition requires that $\epsilon_z = 0$ and

$$\epsilon_\theta + \epsilon_r = \frac{u}{r} + \frac{du}{dr} = 0$$

where u is the radial displacement. From this

$$u = \frac{\sqrt{3}}{2} \frac{c}{r}$$

$$\epsilon_r = -\frac{\sqrt{3}}{2} \frac{c}{r^2}$$

$$\epsilon_\theta = \frac{\sqrt{3}}{2} \frac{c}{r^2}$$

where c is an integration constant and the numerical factor is introduced for convenience. Furthermore, the effective stress becomes

$$\epsilon_e = \frac{c}{r^2}$$

If the stress obtained from the generalized creep law is denoted by $s(\epsilon_e)$ the stress components must obey the relationships

$$\sigma_\theta - \sigma_r = \frac{4}{3} \frac{s(\epsilon_e)}{\epsilon_e} \epsilon_\theta = \frac{2}{\sqrt{3}} s\left(\frac{c}{r^2}\right)$$

and

$$\sigma_z = \frac{1}{2}(\sigma_r + \sigma_\theta)$$

Substituting the above into the equilibrium equation yields

$$d\sigma_r = \frac{2}{\sqrt{3}} \frac{s(c/r^2)}{r} dr = -\frac{1}{\sqrt{3}} \frac{s(v)}{v} dv$$

where $dv = \epsilon_e = c/r^2$. This can be integrated from the condition $\sigma(a) = -p$ to yield

$$\sigma_r = -p - \frac{1}{\sqrt{3}} \int_{v_a}^v \frac{s(v)}{v} dv$$

Expressions for all the stresses can be readily obtained by introducing the boundary condition $\sigma(b) = 0$.

$$\sigma_r = -p - \frac{1}{\sqrt{3}} \int_{v_a}^v \frac{s(v)}{v} dv$$

$$\sigma_\theta = -p - \frac{1}{\sqrt{3}} \int_{v_a}^v \frac{s(v)}{v} dv + \frac{2}{\sqrt{3}} s(v)$$

$$\sigma_z = -p - \frac{1}{\sqrt{3}} \int_{v_a}^v \frac{s(v)}{v} dv + \frac{1}{\sqrt{3}} s(v)$$

If stationary creep is assumed to take place according to the power law $s = \sigma_n(\epsilon/\epsilon_n)^{1/n}$ the above yield

$$\sigma_r = \frac{p}{(b/a)^{2/n} - 1} [1 - (b/r)^{2/n}]$$

$$\sigma_\theta = \frac{p}{(b/a)^{2/n} - 1} [1 - (1 - \frac{2}{n})(b/n)^{2/n}]$$

$$\sigma_z = \frac{p}{(b/a)^{2/n} - 1} [1 - (1 + \frac{1}{n})(b/n)^{2/n}]$$

10.2 Creep in a Rotating Disk

Consider a disk (radius b) of uniform thickness rotating with angular velocity ω . The equilibrium equation is

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r^2 = 0$$

The problem is solved assuming an exponential creep law

$$\dot{\epsilon} = \dot{\epsilon}_e \exp\left(\frac{\sigma}{\sigma_e}\right)$$

Introducing the dimensionless stresses

$$\bar{\sigma}_r = \frac{\sigma_r}{\sigma_e}$$

$$\bar{\sigma}_\theta = \frac{\sigma_\theta}{\sigma_e}$$

and

$$\xi = \frac{r}{b}$$

And the dimensionless creep rates

$$\bar{(\dot{\epsilon}_r)} = \frac{\dot{\epsilon}_r}{\dot{\epsilon}_e} = \frac{d\bar{u}}{d\xi}$$

and

$$\bar{(\dot{\epsilon}_\theta)} = \frac{\dot{\epsilon}_\theta}{\dot{\epsilon}_e} = \frac{\bar{u}}{\xi}$$

where $\bar{u} = u/(b\dot{\epsilon}_e)$ is the dimensionless radial displacement rate.

With the above the equilibrium equation becomes

$$\frac{d}{d\xi}(\xi\bar{\sigma}_r) - \bar{\sigma}_\theta + m\xi^2 = 0$$

where $m = \rho\omega^2 b^2/\sigma_e$.

If there is no applied load at the rim of the disk, $\sigma_r(b) = 0$ and for creep deformation according to the exponential law a characteristic dimensionless parameter α is obtained and is given by

$$\alpha = \sqrt{\frac{2.512}{m}}$$

It can then be shown that two cases are obtained depending on the value of the new parameter. When $\alpha \geq 1$, (i.e. $m \leq 2.512$),

$$\bar{\sigma}_r = \bar{\sigma}_\theta = \frac{1 - \xi^2}{2}$$

throughout the disk.

Alternatively, when $\alpha < 1$, (i.e. $m > 2.512$), two concentric regions form. Inside a circular region of radius $\xi < \alpha$ the stresses are equal to each other and are given by

$$\bar{\sigma}_r = \bar{\sigma}_\theta = C + 1.256\left(1 - \frac{\xi^2}{\alpha^2}\right)$$

where

$$C = \frac{m}{3} - 1 + 0.164\alpha - \ln \alpha$$

In contrast, for radii $\xi > \alpha$ the stresses are different from each other with $\bar{\sigma}_\theta > \bar{\sigma}_r > 0$ and are given by

$$\bar{\sigma}_r = C - \ln \frac{\xi}{\alpha} + \left(1 - \frac{\xi}{\alpha}\right) + 0.836\left(\frac{\alpha}{\xi} - \frac{\xi^2}{\alpha^2}\right)$$

and

$$\bar{\sigma}_\theta = C - \ln \frac{\xi}{\alpha}$$

Further, the creep strain rates are

$$\bar{(\dot{\epsilon}_r)} = 0$$

and

$$\bar{(\dot{\epsilon}_\theta)} = \frac{\alpha}{\xi} \exp(C)$$

It is easy to show that stresses and displacements are continuous at $\xi = \alpha$.

The stresses for the rotating disk with a hole at the center are determined as above. In this case, the dimensionless inner radius is taken as $\alpha = a/b$. The stresses in the disk are given by

$$\bar{\sigma}_r = C_1 + \ln \frac{\alpha}{\xi} + 1 - \frac{m\xi^2}{3} - \frac{C_2}{\xi}$$

and

$$\bar{\sigma}_\theta = C_1 + \ln \frac{\alpha}{\xi}$$

where

$$C_1 = \frac{m(1 + \alpha + \alpha^2)}{3(1 - \alpha)} - 1 - \frac{\ln \alpha}{1 - \alpha}$$

and

$$C_2 = \frac{m\alpha(1 + \alpha)}{3} - \frac{\alpha \ln \alpha}{1 - \alpha}$$

11 Exercises

11.1

A titanium component is going to be subjected to a temperature of $T = 1000K$ and a stress $\sigma = 2.5 \times 10^6$ Pa, for a long time and a life estimate is required. An accelerated creep stress rupture test is conducted to determine a steady state creep rate of $\dot{\epsilon} = 10^{-3}s^{-1}$ and a corresponding rupture time of 0.1 hours.

a) Determine the constant in the Monkman-Grant equation.

b) Assume dislocation creep is the dominant creep mechanism under the stated exposure conditions, i.e. the steady state creep rate can be estimated using the Dorn equation,

$$\dot{\epsilon} = \frac{AGb}{kT} D \left(\frac{\sigma}{G} \right)^n$$

where the values of the parameters at the stated temperature are $A = 7.7 \times 10^4$; $G = 2.5 \times 10^{10}$ Pa; $b = 3 \times 10^{-10}$ m; $D = 5 \times 10^{-11} m^2/s$; $n = 4.3$; Moreover $k = 1.381 \times 10^{-23} J/K$.

Determine the value of $\dot{\epsilon}$ under the stated service conditions and then use the Monkman-Grant equation again to estimate the rupture life of the component under those conditions.

11.2

Consider a long, thick walled (inner radius a , outer radius $b = 2a$) pressurized tube. When the pressurized tube is exposed to high temperatures, creep occurs. Assuming power law creep with exponent n , the stresses are given by

$$\sigma_r = \frac{p}{\left(\frac{b}{a}\right)^{2/n} - 1} \left[1 - \left(\frac{b}{r}\right)^{2/n} \right]$$

and

$$\sigma_\theta = \frac{p}{\left(\frac{b}{a}\right)^{2/n} - 1} \left[1 - \left(1 - \frac{2}{n}\right) \left(\frac{b}{r}\right)^{2/n} \right]$$

Rearrange the equations into dimensionless form and then plot the ratios σ_r/p and σ_θ/p versus the ratio r/b , for the following values of the power law parameter, $n = 1; n = 2; n = 5; n = 8$.

11.3

Use the generic creep law

$$\sigma = K\epsilon^{1/M}\dot{\epsilon}^{1/N}$$

to produce systems of creep curves for IN 100 alloy for stresses of 160 and 180 MPa and temperatures of 800 and 1000 degrees Celsius. Use values for the parameters in the equation as given in the table below. Determine and provide the strain rate for the secondary creep stage in each case.

T (C)	N	M	K(MPas)
800	17	13.2	2140
1000	10.8	9.8	1450

11.4

A cantilever beam has length $L = 1$, breadth $b = 1$ and height $h = 0.1$ (all in meters). The beam is maintained at a uniform temperature of 1250 degrees Celsius while at the lower edge of the free end, a downward force P is applied. Use the Ansys finite element software with linear elements to determine approximate solutions to this problem. Investigate the creep deformation of the beam as a function of the applied load P assuming the Bailey-Norton law

$$\epsilon^c = A\sigma^n t^m$$

is applicable, where the stress is in Pa and time in seconds. The values of the parameters are (SI units) $A = 1.523 \times 10^{-17} \exp(-28,392/(T + 273))$, $n = 3.15$, and $m = 1$. Assume the elastic modulus is constant and equal to $E = 1.9 \times 10^{10}$ Pa Start your analysis using small values of P .